Exercise 2, Chapter 4. (Math 414-501, Spring 2020)

The function $f(x)$ is given by

$$f(x) = \begin{cases} 2, & 0 \le x < 1/4, \\ -3, & 1/4 \le x < 1/2, \\ 1, & 1/2 \le x < 3/4, \\ 3, & 3/4 \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since f is in V_2, we can write in terms of the basis $\{\phi(2^2x - k)\}_{k=0}^3$ (cf. Definition 4.1 in the text):

$$f(x) = 2\phi(4x) - 3\phi(4x-1) + \phi(4x-2) + 3\phi(4x-3).$$

There are two ways to do this problem. The first way to approach decomposing f into its components along V_0, W_0, and W_1 is to use Lemma 4.10, which states that

$$\begin{aligned} \phi(2^j x) &= (\phi(2^{j-1}x) + \psi(2^{j-1}x))/2 \\ \phi(2^j x - 1) &= (\phi(2^{j-1}x) - \psi(2^{j-1}x))/2. \end{aligned}$$

Begin by getting the V_1, W_1 parts. To do this, replace the functions $\phi(4x-k)$ as follows:

$$\begin{aligned} \phi(4x) &= (\phi(2x) + \psi(2x))/2, \\ \phi(4x-1) &= (\phi(2x) - \psi(2x))/2, \\ \phi(4x-2) &= (\phi(2x-1) + \psi(2x-1))/2, \\ \phi(4x-3) &= (\phi(2x-1) - \psi(2x-1))/2 \end{aligned}$$

Using these, put f into the form

$$\begin{aligned} f(x) &= (1-3/2)\phi(2x) + (1+3/2)\psi(2x) + (1/2+3/2)\phi(2x-1) + (1/2-3/2)\psi(2x-1) \\ &= \underbrace{-\frac{1}{2}\phi(2x) + 2\phi(2x-1)}_{V_1} + \underbrace{\frac{5}{2}\psi(2x) - \psi(2x-1)}_{W_1}. \end{aligned}$$

Finally, use $\phi(2x) = (\phi(x) + \psi(x))/2$ and $\phi(2x-1) = (\phi(x) - \psi(x))/2$ in the equation above to obtain the desired decomposition:

$$f(x) = \underbrace{\frac{3}{4}\phi(x)}_{V_0} \underbrace{-\frac{5}{4}\psi(x)}_{W_0} + \underbrace{\frac{5}{2}\psi(2x) - \psi(2x-1)}_{W_1}.$$

There is a second method for solving this problem. The $j = 2$ level coefficients for f can be put in row vector form: $a^2 = [2 \quad -3 \quad 1\,3\,]$, where the first entry is a_0^2, the last is a_3^2. All other a_k^2 coefficients are 0.

The first step in the decomposition is to find the $j = 1$ level coefficients; that is, the a_k^1's and b_k^1. To do this, use the formulas below, which are found in Theorem 4.12.

$$a_k^{j-1} = \frac{a_{2k}^j + a_{2k+1}^j}{2} \quad \text{and} \quad b_k^{j-1} = \frac{a_{2k}^j - a_{2k+1}^j}{2}.$$

In our case, we have $j = 2$. Our aim is find a^0, b^1 and b^0; these will give us the breakdown into V_0, W_0 and W_1.

We start by finding a^1. Note that both $a_{2k}^2 = 0$ and $a_{2k+1}^2 = 0$ for $k < 0$ and $k > 1$. It follows that we only need to find a_k^1 and b_k^1 for $k = 0, 1$. Using the formulas above we obtain

$$a_0^1 = -\frac{1}{2},\ a_1^1 = \frac{4}{2} = 2$$

so that $a^1 = [-\frac{1}{2}\ 2\,]$. Similarly, $b^1 = [\frac{5}{2}\ -1\,]$. Following the procedure above for a^0, we see that $a_0^0 = \frac{3}{4}$; thus, $a^0 = [\frac{3}{4}\,]$. Similarly, $b_0^0 = \frac{-\frac{1}{2}-2}{2} = -\frac{5}{4}$. Thus, $b^0 = [-\frac{5}{4}\,]$. Finally, we have $f(x) = \frac{3}{4}\phi(x) - \frac{5}{4}\psi(x) + \frac{5}{2}\psi(2x) - \psi(2x - 1)$, which agrees with the answer from the previous method.