Compact Sets and Compact Operators
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Throughout these notes, $\mathcal{H}$ denotes a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$. We also note that $\mathcal{B}(\mathcal{H})$ is a Banach space under the usual operator norm. (See problem 6(a) on the Final Exam.)

1 Compact and Precompact Subsets of $\mathcal{H}$

Definition 1.1. A subset $S$ of $\mathcal{H}$ is said to be compact if and only if it is closed and every sequence in $S$ has a convergent subsequence. $S$ is said to be precompact if its closure is compact.

Proposition 1.2. Here are some important properties of compact sets.

1. Every compact set is bounded.

2. A bounded set $S$ is precompact if and only if every bounded sequence has a convergent subsequence.

3. Let $\mathcal{H}$ be finite dimensional. Every closed, bounded subset of $\mathcal{H}$ is compact.

4. In an infinite dimensional space, closed and bounded is not enough.

Proof. Properties 2 and 3 are left to the reader. For property 1, assume that $S$ is an unbounded compact set. Since $S$ is unbounded, we may select a sequence $\{v_n\}_{n=1}^\infty$ such that $\|v_n\| \to 0$ as $n \to \infty$. Since $S$ is compact, this sequence will have a convergent subsequence, say $\{v_k\}_{k=1}^\infty$, which will still be unbounded. This sequence is Cauchy, so there is a positive integer $K$ for which $\|v_\ell - v_m\| \leq 1/2$ for all $\ell, m \geq K$. Fix $\ell$ and note that by the triangle inequality $\|v_m\| \leq 1/2 + \|v_\ell\|$. Now, the right side is bounded, because $\ell$ is fixed. However, $\|v_m\| \to \infty$ as $m \to \infty$. This is a contradiction, so $S$ must be bounded. For property 4, let $S = \{f \in \mathcal{H} : \|f\| \leq 1\}$. Every o.n. basis $\{\phi_n\}_{n=1}^\infty$ is in $S$. However, for such a basis $\|\phi_m - \phi_n\| = \sqrt{2}$, $n \neq m$. Again, this means there are no Cauchy subsequences in $\{\phi_n\}_{n=1}^\infty$, and consequently, no convergent subsequences. Thus, $S$ is not compact. \qed
2 Compact Operators

Definition 2.1. Let $K : \mathcal{H} \to \mathcal{H}$ be linear. $K$ is said to be compact if and only if $K$ maps bounded sets into precompact sets. Equivalently, if $\{v_n\}_{n=1}^{\infty}$ is bounded, then the sequence $\{Kv_n\}_{n=1}^{\infty}$ has a convergent subsequence. We denote the set of compact operators on $\mathcal{H}$ by $\mathcal{C}(\mathcal{H})$.

Proposition 2.2. If $K \in \mathcal{C}(\mathcal{H})$, then $K$ is bounded – i.e., $\mathcal{C}(\mathcal{H}) \subset B(\mathcal{H})$. In addition, $\mathcal{C}(\mathcal{H})$ is a subspace of $B(\mathcal{H})$.

Proof. We leave this as an exercise for the reader. \qed

We now turn to giving some examples of compact operators. We start with the finite-rank operators. If the range of an operator $K$ is finite dimensional, then we say that $K$ is a finite-rank operator. For bounded set $B \in \mathcal{H}$, the $K|_S$ is a bounded subset of a finite dimensional space and is therefore precompact. It follows that $K$ is in $\mathcal{C}(\mathcal{H})$.

To describe $K$ explicitly, let $\{\phi_k\}_{k=1}^{n}$ be a basis for $\mathbb{R}(K)$. Then, $Kf = \sum_{k=1}^{n} a_k \phi_k$. We want to see how the $a_k$’s depend on $f$. Consider $\langle Kf, \phi_j \rangle = \langle f, K^* \phi_j \rangle = \sum_{k=1}^{n} a_k \langle \phi_k, \phi_j \rangle$. Next let $\psi_j = K^* \phi_j$, so that $\langle f, K^* \phi_j \rangle = \langle f, \psi_j \rangle$. Because $\{\phi_k\}_{k=1}^{n}$ is a basis, it is linear independent. Hence, the Gram matrix $G_{j,k} = \langle \phi_k, \phi_j \rangle$ is invertible, and so we can solve the system of equations $\langle f, \psi_j \rangle = \sum_{k=1}^{n} G_{j,k} a_k$. Doing so yields $a_k = \sum_{j=1}^{n} (G^{-1})_{k,j} \langle f, \psi_j \rangle$. The $a_k$’s are obviously linear in $f$. Of course, a different basis will give a different representation.

Let $\mathcal{H} = L^2[0,1]$. A particularly important set of finite rank operators in $\mathcal{C}(\mathcal{H})$ are ones given by finite rank or degenerate kernels, $k(x, y) = \sum_{k=1}^{n} \phi_k(x) \overline{\psi_k(y)}$, where the functions involved are in $L^2$. The operator is then $Kf(x) = \int_0^1 k(x, y) f(y) dy$. In the example that we did for resolvents, the kernel was $k(x, y) = x^2 y$, and the operator was $Ku(x) = \int_0^1 k(x, y) u(y) dy$. We will show that the Hilbert-Schmidt kernels also yield compact operators. This will follow as a corollary to our next theorem, which is especially important.

Theorem 2.3. $\mathcal{C}(\mathcal{H})$ is a closed subspace of $B(\mathcal{H})$.

Proof. Suppose that $\{K_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{C}(\mathcal{H})$ that converges to $K \in B(\mathcal{H})$, in the operator norm. We want to show that $K$ is compact. Assume the $\{v_k\}$ is a bounded sequence in $\mathcal{H}$, with $\|v_k\| \leq C$ for all $k$. Compactness will follow if we can prove that $\{Kv_k\}$ has a convergent subsequence. The technique for doing this is often called a diagonalization argument. We start with the full sequence and form $\{K_1 v_k\}$. Since $K_1$ is compact, we can
select a subsequence \( \{ v_k^{(1)} \} \) such that \( \{ K_1 v_k^{(1)} \} \) is convergent. We may carry out the same procedure with \( \{ K_2 v_k^{(1)} \} \), selecting a subsequence of \( \{ K_2 v_k^{(1)} \} \) that is convergent. Call it \( \{ v_k^{(2)} \} \). Since this is a subsequence of \( \{ v_k^{(1)} \} \), \( \{ K_1 v_k^{(2)} \} \) is convergent. Continuing in this way, we construct subsequences \( \{ v_k^{(j)} \} \) for which \( \{ K_m v_k^{(j)} \} \) is convergent for all \( 1 \leq m \leq j \). Next, we let \( \{ u_j := v_j^{(i)} \} \), the “diagonal” sequence. This is a subsequence of all of the \( \{ v_k^{(j)} \} \)’s. Consequently, for \( n \) fixed, \( \{ K_n u_j \}_{j=1}^{\infty} \) will be convergent. To finish up, we will use an “up, over, and around” argument. Note that for all \( \ell, m, \)

\[
\| K u_\ell - K u_m \| \leq \| K u_\ell - K u_\ell u_\ell \| + \| K u_\ell u_\ell - K u_m \| + \| K u_m - K u_m \| 
\]

Since \( \| K u_\ell - K u_\ell u_\ell \| \leq \| K - K \|_{op} \| u_\ell \| \leq 2C \| K - K \|_{op} \) and, similarly, \( \| K u_m - K u_m \| \leq 2C \| K - K \|_{op} \), so we have \( \| K u_\ell - K u_m \| \leq 4C \| K - K \|_{op} + \| K u_\ell - K u_m \| \). Let \( \varepsilon > 0 \). First choose \( N \) such that for \( n \geq N \), \( \| K - K \|_{op} < \varepsilon/(8C) \). Fix \( n \). Because \( \{ K_n u_\ell \} \) is convergent, it is Cauchy. Choose \( N' \) so large that \( \| K_n u_\ell - K_n u_m \| < \varepsilon/2 \) for all \( \ell, m \geq N' \). Putting these two together yields \( \| K u_\ell - K u_\ell \| \leq \varepsilon \), provided \( \ell, m \geq N' \). Thus \( \{ K u_\ell \} \) is Cauchy and therefore convergent.

**Corollary 2.4.** Hilbert-Schmidt operators are compact.

**Proof.** Let \( \mathcal{H} = L^2[0,1] \) and suppose \( k(x, y) \in L^2(R) \), \( R = [0,1] \times [0,1] \). The associated Hilbert-Schmidt operator is \( Ku = \int_0^1 k(x,y)u(y)dy \). Let \( \{ \phi_n \}_{n=1}^{\infty} \) be an o.n basis for \( L^2[0,1] \). With a little work, one can show that \( \{ \phi_n(x)\phi_m(y) \}_{n,m=1}^{\infty} \) is an o.n basis for \( L^2(R) \). Also, from example 2 in the notes on Bounded Operators (11/7/13), we have that \( \| K \|_{op} \leq \| k \|_{L^2(R)} \). Expand \( k(x, y) \) in the o.n basis \( \{ \phi_n(x)\phi_m(y) \}_{n,m=1}^{\infty} \):

\[
k(x,y) = \sum_{n,m=1}^{\infty} \alpha_{n,m} \phi_n(x)\phi_m(y), \quad \alpha_{n,m} = \langle k(x,y), \phi_n(x)\phi_m(y) \rangle_{L^2(R)}
\]

Next, let \( k_N(x, y) = \sum_{n,m=1}^{N} \alpha_{n,m} \phi_n(x)\phi_m(y) \) and also \( K_N \) be the finite rank operator \( K_N u(x) = \int_0^1 k_N(x,y)u(y)dy \). By Parseval’s theorem, we have that \( \| k - k_N \|_{L^2(R)}^2 = \sum_{n,m=1}^{\infty} |\alpha_{n,m}|^2 \) and by example 2 mentioned above, \( \| K - K_N \|_{op} \leq \| k - k_N \|_{L^2(R)} \), so

\[
\| K - K_N \|_{op}^2 \leq \sum_{n,m=N+1}^{\infty} |\alpha_{n,m}|^2 
\]

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Because the series on the right above converges to 0 as \( N \to \infty \), we have \( \lim_{N \to \infty} \| K - K_N \| = 0 \). Thus \( K \) is the limit in \( \mathcal{B}(L^2[0, 1]) \) of finite rank operators, which are compact. By the theorem above, \( K \) is also compact. □

We now turn to some of the algebraic properties of \( \mathcal{C}(\mathcal{H}) \).

**Proposition 2.5.** Let \( K \in \mathcal{C}(\mathcal{H}) \) and let \( L \in \mathcal{B}(\mathcal{H}) \). Then both \( KL \) and \(LK\) are in \( \mathcal{C}(\mathcal{H}) \).

**Proof.** Let \( \{v_k\} \) be a bounded sequence in \( \mathcal{H} \). Since \( L \) is bounded, the sequence \( \{L v_k\} \) is also bounded. Because \( K \) is compact, we may find a subsequence of \( \{KLv_k\} \) that is convergent, so \( KL \in \mathcal{C}(\mathcal{H}) \). Next, again assuming \( \{v_k\} \) is a bounded sequence in \( \mathcal{H} \), we may extract a convergent subsequence from \( \{Kv_k\} \), which, with a slight abuse of notation, we will denote by \( \{Kv_j\} \). Because \( L \) is bounded, it is also continuous. Thus \( \{LKv_j\} \) is convergent. It follows that \( LK \) is compact. □

**Proposition 2.6.** \( K \) is compact if and only if \( K^* \) is compact.

**Proof.** Because \( K \) is compact, it is bounded and so is its adjoint \( K^* \), in fact \( \|K^*\|_{op} = \|K\|_{op} \). By Proposition 2.5, we thus have that \( KK^* \) is compact. It follows that if \( \{u_n\} \) be a bounded sequence in \( \mathcal{H} \), then we may extract a subsequence \( \{u_j\} \) such that the sequence \( \{KK^*u_j\} \) is convergent. This of course means that this sequence is also Cauchy. Note that

\[
\langle KK^*(v_j - v_k), v_j - v_k \rangle = \langle K^*(v_j - v_k), K^*(v_j - v_k) \rangle = \|K^*(v_j - v_k)\|^2.
\]

From and the fact that \( \{v_j\} \) is bounded, we see that \( \langle KK^*(v_j - v_k), v_j - v_k \rangle \leq \|v_j - v_k\| \|KK^*(v_j - v_k)\| \leq C\|KK^*(v_j - v_k)\| \). Thus,

\[
\|K^*(v_j - v_k)\|^2 \leq C\|KK^*(v_j - v_k)\|,
\]

Since \( \{KK^*v_j\} \) is Cauchy, for every \( \varepsilon > 0 \), we can find \( N \) such that whenever \( j, k \geq N \), \( \|KK^*(v_j - v_k)\| < \varepsilon^2 / C \). It follows that \( \|K^*(v_j - v_k)\| < \varepsilon \), if \( j, k \geq N \). This implies that \( \{K^*v_j\} \) is Cauchy and therefore convergent. □

We want to put this in more algebraic language. Taking \( L \) to be compact in Proposition 2.5, we have that the product of two compact operators is compact. Since \( \mathcal{C}(\mathcal{H}) \) is already a subspace, this implies that it is an algebra. Moreover, by taking \( L \) to be just a bounded operator, we have that \( \mathcal{C}(\mathcal{H}) \) is a two-sided \emph{ideal} in the algebra \( \mathcal{B}(\mathcal{H}) \). Since \( K \) being compact implies \( K^* \) is compact, \( \mathcal{C}(\mathcal{H}) \) is closed under the operation of taking adjoints; thus, \( \mathcal{C}(\mathcal{H}) \) is a \( * \)-ideal. Finally, including the result of Theorem 2.3, we have that \( \mathcal{C}(\mathcal{H}) \) is a closed under limits. We summarize these results as follows.

**Theorem 2.7.** \( \mathcal{C}(\mathcal{H}) \) is a closed, two-sided, \( * \)-ideal in \( \mathcal{B}(\mathcal{H}) \).