X-ray Tomography & Integral Equations
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X-ray Tomography. An important part of X-ray tomography – the CAT scan – is solving a mathematical problem that goes back to the earlier twentieth century work of the mathematician Johann Radon: Suppose that there is a function $f(x,y)$ defined in a region of the plane and that all we know about $f$ is the collection of line integrals $\int_L f(x(s),y(s))ds$ over each line $L$ that intersects the region. (See Figure. 1.) The problem is to find $f$, given this information.

![Diagram of X-ray tomography](image)

Figure 1: The region where $f$ is defined and a typical line $L$ cutting the region are shown. $L$ is specified by $\rho$ and the angle $\theta$.

We will assume that the region is a disk $D := \{|x| \leq 1\}$. The unit vector $n$ that is normal to $L$ and points away from the origin is $n = \cos(\theta)i + \sin(\theta)j$. The tangent\(^1\) pointing upward is $t = -\sin(\theta)i + \cos(\theta)j$. If we let $s \geq 0$ be

\(^1\)In class we used $\varphi$ instead of $\theta$. 

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the arc length starting at the point $\rho \mathbf{n}$, then any point $\mathbf{x}$ above $\rho \mathbf{n}$ is specified by $\mathbf{x} = st + \rho \mathbf{n}$. If $\mathbf{x}$ is below $\rho \mathbf{n}$, then it is specified by $\mathbf{x} = -st + \rho \mathbf{n}$.

We will work with $\mathbf{x}$ above $\rho \mathbf{n}$. Express $\mathbf{x}$ in terms of polar coordinates $(r, \phi)$, $\mathbf{x} = r \cos(\phi) \mathbf{i} + r \sin(\phi) \mathbf{j}$. Of course, $r = |\mathbf{x}|$. Comparing this with $\mathbf{x} = st + \rho \mathbf{n}$, we see that $r^2 = s^2 + \rho^2$ and $\rho = \mathbf{x} \cdot \mathbf{n} = r \cos(\phi - \theta)$. Since $\mathbf{x}$ is above $\rho \mathbf{n}$, we have that $\phi \geq \theta$ and thus $\phi = \theta + \cos^{-1}(\rho/r)$.

When $\mathbf{x}$ is below $\rho \mathbf{n}$, $\phi \leq \theta$ and $\phi = \theta - \cos^{-1}(\rho/r)$. Breaking the integral $\int_L f(\mathbf{x}(s))ds$ into two pieces, making the change of variables $s = \sqrt{r^2 - \rho^2}$; $ds = (r^2 - \rho^2)^{-1/2}r dr$, and noting that $\rho \leq r \leq 1$, we have

$$
\int_L f(\mathbf{x}(s))ds = \int_{\phi \geq \theta} f(\mathbf{x}(s))ds + \int_{\phi \leq \theta} f(\mathbf{x}(s))ds
= \int_{\rho}^{1} \frac{f(r, \theta + \cos^{-1}(\rho/r))dr}{\sqrt{(r^2 - \rho^2)}} + \int_{\rho}^{1} \frac{f(r, \theta - \cos^{-1}(\rho/r))dr}{\sqrt{(r^2 - \rho^2)}}
= \int_{\rho}^{1} \frac{(f(r, \theta + \cos^{-1}(\rho/r)) + f(r, \theta - \cos^{-1}(\rho/r)))dr}{\sqrt{(r^2 - \rho^2)}}.
$$

Assuming the $f(\mathbf{x}) = f(r, \phi)$ is smooth enough, we can expand it in a Fourier series in $\phi$,$$
f(r, \phi) = \sum_{n=-\infty}^{\infty} \hat{f}_n(r)e^{in\phi},$$
and then replace $f$ in the integral on the right above by this series. Again making the assumption that interchanging sum and integral is possible and manipulating the resulting expression, we have

$$
\int_L f(\mathbf{x}(s))ds = 2 \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{\rho}^{1} \hat{f}_n(r) \frac{\cos(n \cos^{-1}(\rho/r))rdr}{\sqrt{r^2 - \rho^2}}. \quad (1)
$$

Since the line $L$ is specified by the angle $\theta$ and distance $\rho$, the integral over $L$ is a function of $\theta$ and $\rho$, which we denote by $F(\rho, \theta)$. In addition, the expression $T_n(\rho/r) := \cos(n \cos^{-1}(\rho/r))$ is actually an $n$th degree Chebyshev polynomial. For example, $T_2(\rho/r) = 2 \cos^2(\cos^{-1}(\rho/r)) - 1 = 2(\rho/r)^2 - 1$. Using these two facts in connection with (1) we have

$$
F(\rho, \theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{\rho}^{1} \hat{f}_n(r) \frac{T_n(\rho/r)r}{\sqrt{r^2 - \rho^2}}dr. \quad (2)
$$
The Fourier series for $F(\rho, \theta) = \sum_{n=-\infty}^{\infty} \hat{F}_n(\rho)e^{in\theta}$. Comparing it with the series in (2) we arrive at

$$\hat{F}_n(\rho) = \int_{\rho}^{1} \hat{f}_n(r) \frac{T_n(\rho/r)r}{\sqrt{r^2 - \rho^2}} dr, \ n \in \mathbb{Z}. \quad (3)$$

The point is that $F(\rho, \theta) = \int_L f(x(s))ds$ is known, and so the Fourier coefficients $\hat{F}_n(\rho)$ are all known. The problem of finding $f$, given $F$, is thus equivalent to solving the integral equations in (3) for the $\hat{f}_n(r)$’s and recovering $f(r, \phi)$ from its Fourier series.

**Classification of integral equations.** Certain types of integral equations come up often enough that they are grouped into classes, which are described below. There, the function $f$ and kernel $k(x, y)$ are known, $u$ is the unknown function to be solved for, and $\lambda$ is a parameter. The integral equations in (3) are Volterra equations of the first kind.

**Fredholm Equations**

- **1st kind.** $f(x) = \int_{a}^{b} k(x, y)u(y)dy$.
- **2nd kind.** $u(x) = f(x) + \lambda \int_{a}^{b} k(x, y)u(y)dy$.

**Volterra Equations**

- **1st kind.** $f(x) = \int_{a}^{x} k(x, y)u(y)dy$.
- **2nd kind.** $u(x) = f(x) + \lambda \int_{a}^{x} k(x, y)u(y)dy$.

**Acknowledgments** Figure 1 is from the article “A small note on Matlab iradon and the all-at-once vs. the one-at-a-time method,” by Nasser M. Abbasi. July 17, 2008. The figure was downloaded on November 10, 2013, from the website

http://12000.org/my_notes/note_on_radon/
note_on_radon/note_on_radon.htm