Compact Sets and Compact Operators
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Throughout these notes, $\mathcal{H}$ denotes a separable Hilbert space. We will use the notation $\mathcal{B}(\mathcal{H})$ to denote the set of bounded linear operators on $\mathcal{H}$. We also note that $\mathcal{B}(\mathcal{H})$ is a Banach space, under the usual operator norm.

1 Compact and Precompact Subsets of $\mathcal{H}$

Definition 1.1. A subset $S$ of $\mathcal{H}$ is said to be compact if and only if it is closed and every sequence in $S$ has a convergent subsequence. $S$ is said to be precompact if its closure is compact.

Proposition 1.2. Here are some important properties of compact sets.

1. Every compact set is bounded.

2. Let $S$ be a bounded set. Then $S$ is precompact if and only if every sequence has a convergent subsequence.

3. Let $\mathcal{H}$ be finite dimensional. Every closed, bounded subset of $\mathcal{H}$ is compact.

4. In an infinite dimensional space, closed and bounded is not enough.

Proof. Properties 2 and 3 are left to the reader. For property 1, assume that $S$ is an unbounded compact set. Since $S$ is unbounded, we may select a sequence $\{v_n\}_{n=1}^{\infty}$ from $S$ such that $\|v_n\| \to \infty$ as $n \to \infty$. Since $S$ is compact, this sequence will have a convergent subsequence, say $\{v_{n_k}\}_{k=1}^{\infty}$, which still will be unbounded.(Why?) Let $v = \lim_{k \to \infty} v_{n_k}$. Thus, for $\varepsilon = 1$ there is a positive integer $K$ for which $\|v - v_{n_k}\| < 1$ for all $k \geq K$. By the triangle inequality, $\|v_{n_k}\| \leq \|v\| + 1$. Now, the right side is bounded, but the left side isn’t, since $\|v_{n_k}\| \to \infty$ as $k \to \infty$. This is a contradiction, so $S$ must be bounded. For property 4, let $S = \{f \in \mathcal{H}: \|f\| \leq 1\}$. Every o.n. basis $\{\phi_n\}_{n=1}^{\infty}$ is in $S$. However, for such a basis $\|\phi_m - \phi_n\| = \sqrt{2}$, $n \neq m$. Again, this means there are no Cauchy subsequences in $\{\phi_n\}_{n=1}^{\infty}$, and consequently, no convergent subsequences. Thus, $S$ is not compact. \[ \square \]
2 Compact Operators

Definition 2.1. Let $K : \mathcal{H} \rightarrow \mathcal{H}$ be linear. $K$ is said to be compact if and only if $K$ maps bounded sets into precompact sets. Equivalently, $K$ is compact if and only if for every bounded sequence $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ the sequence $\{KV_n\}_{n=1}^{\infty}$ has a convergent subsequence. We denote the set of compact operators on $\mathcal{H}$ by $C(\mathcal{H})$.

Proposition 2.2. If $K \in C(\mathcal{H})$, then $K$ is bounded – i.e., $C(\mathcal{H}) \subset B(\mathcal{H})$. In addition, $C(\mathcal{H})$ is a subspace of $B(\mathcal{H})$.

Proof. We leave this as an exercise for the reader.

We now turn to giving some examples of compact operators. We start with the finite-rank operators. If the range of a bounded operator $K$ is finite dimensional, then we say that $K$ is a finite-rank operator.

Proposition 2.3. Every finite-rank operator $K$ is compact.

Proof. The range of $K$ is finite dimensional, so every bounded subset of the range is precompact. Let $S \subseteq \{f \in \mathcal{H} : \|f\| \leq C\}$, where $C$ is fixed. Note that the range of $K$ restricted to $S$ is also bounded: $\|Kf\| \leq \|K\|_{op}\|f\| \leq C\|K\|_{op}$. Thus, $K$ maps a bounded set $S$ into a bounded subset of a finite dimensional subspace of $\mathcal{H}$, which is itself precompact. Hence, $K$ is thus compact.

To describe $K$ explicitly, let $\{\phi_k\}_{k=1}^{n}$ be a basis for $R(K)$. Then, $Kf = \sum_{k=1}^{n} a_k \phi_k$. We want to see how the $a_k$’s depend on $f$. Consider $\langle Kf, \phi_j \rangle = \langle f, K^* \phi_j \rangle = \sum_{k=1}^{n} a_k \langle \phi_k, \phi_j \rangle$. Next let $\psi_j = K^* \phi_j$, so that $\langle f, K^* \phi_j \rangle = \langle f, \psi_j \rangle$. Because $\{\phi_k\}_{k=1}^{n}$ is a basis, it is linear independent. Hence, the Gram matrix $G_{j,k} = \langle \phi_k, \phi_j \rangle$ is invertible, and so we can solve the system of equations $\langle f, \psi_j \rangle = \sum_{k=1}^{n} G_{j,k} a_k$. Doing so yields $a_k = \sum_{j=1}^{n} (G^{-1})_{k,j} \langle f, \psi_j \rangle$. The $a_k$’s are obviously linear in $f$. Of course, a different basis will give a different representation.

Let $\mathcal{H} = L^2[0,1]$. A particularly important set of finite rank operators in $C(\mathcal{H})$ are ones given by finite rank or degenerate kernels, $k(x, y) = \sum_{k=1}^{n} \phi_k(x)\psi_k(y)$, where the functions involved are in $L^2$. The operator is then $Kf(x) = \int_{0}^{1} k(x,y)f(y)dy$. In the example that we did for resolvents, the kernel was $k(x, y) = xy^2$, and the operator was $Ku(x) = \int_{0}^{1} k(x,y)u(y)dy$. Later, we will show that the Hilbert-Schmidt kernels also yield compact operators. Before, we do so, we will discuss a few more properties of compact operators.
Lemma 2.4. Let \( \{ \phi_n \}_{n=1}^{\infty} \) be an o.n. set in \( \mathcal{H} \) and let \( K \in \mathcal{C}(\mathcal{H}) \). Then, 
\[
\lim_{n \to \infty} K\phi_n = 0.
\]

Proof. Suppose not. Then we may select a subsequence \( \{ \phi_m \} \) of \( \{ \phi_n \}_{n=1}^{\infty} \) for which \( \|K\phi_m\| \geq \alpha > 0 \) for all \( m \). Because \( K \) is compact, we can also select a subsequence \( \{ \phi_k \} \) of \( \{ \phi_m \} \) for which \( K\phi_k \) is convergent to \( \psi \in \mathcal{H} \). Now, \( \{ \phi_k \} \) being a subsequence of \( \{ \phi_m \} \) implies that \( \|K\phi_k\| \geq \alpha > 0 \). Taking the limit in this inequality yields \( \|\psi\| \geq \alpha > 0 \). Next, note that\( \lim_{k \to \infty} \langle K\phi_k, \psi \rangle = \|\psi\|^2 \). However, \( \lim_{k \to \infty} \langle K\phi_k, \psi \rangle = \lim_{k \to \infty} \langle \phi_k, K^*\psi \rangle = 0 \), by Bessel’s inequality. Thus, \( \|\psi\|^2 = 0 \), which is a contradiction. \( \square \)

This lemma is a special case of a more general result. We say that a sequence \( \{ f_n \} \) is weakly convergent to a \( f \in \mathcal{H} \) if and only if for all \( g \in \mathcal{H} \) we have \( \lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle \). For example, the o.n. set in the lemma weakly converges to 0.

There are two important facts concerning weak convergence\( ^1 \). The first is that weakly convergent sequences are bounded and the second is that every bounded sequence has a weakly convergent subsequence.

Proposition 2.5. Let \( \{ f_n \} \) weakly converge to \( f \in \mathcal{H} \). If \( K \in \mathcal{C}(\mathcal{H}) \), then \( \lim_{n \to \infty} Kf_n = Kf \). That is, \( K \) maps weakly convergent sequences into “strongly” convergent ones.

Proof. The proof is similar to that of Lemma 2.4. Suppose not. Then there exists \( \epsilon > 0 \) and a subsequence \( \{ f_{n_k} \} \) such that \( \|Kf_{n_k} - Kf\| \geq \epsilon > 0 \). Because \( K \in \mathcal{C}(\mathcal{H}) \), we may select a subsequence of \( \{ f_{n_k} \} \), \( f_{n_k} =: f_j \), such that \( Kf_j \) converges to \( \psi \). From the inequality above, we have that \( \|\psi - Kf\| \geq \epsilon \). We can use this and the weak convergence of \( Kf_j \) to arrive at a contradiction. We leave the details as an exercise. \( \square \)

We remark that the converse is true, too. This leads to an alternative characterization of compact operators: \( K \) is compact if and only if \( K \) maps weakly convergent sequences into strongly convergent ones. See the book "Functional Analysis," by F. Riesz and B. Sz.-Nagy.

Our next result is one of the most important theorems in the theory of compact operators.

Theorem 2.6. \( \mathcal{C}(\mathcal{H}) \) is a closed subspace of \( \mathcal{B}(\mathcal{H}) \).

\(^1\) See Riesz-Nagy, p. 64.
Proof. Suppose that $\{K_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{C}(\mathcal{H})$ that converges to $K \in \mathcal{B}(\mathcal{H})$, in the operator norm. We want to show that $K$ is compact. Assume the $\{v_k\}$ is a bounded sequence in $\mathcal{H}$, with $\|v_k\| \leq C$ for all $k$. Compactness will follow if we can prove that $\{Kv_k\}$ has a convergent subsequence. The technique for doing this is often called a diagonalization argument. We start with the full sequence and form $\{K_1v_k\}$. Since $K_1$ is compact, we can select a subsequence $\{v_k^{(1)}\}$ such that $\{K_1v_k^{(1)}\}$ is convergent. We may carry out the same procedure with $\{K_2v_k^{(1)}\}$, selecting a subsequence of $\{K_2v_k^{(1)}\}$ that is convergent. Call it $\{v_k^{(2)}\}$. Since this is a subsequence of $\{v_k^{(1)}\}$, $\{K_1v_k^{(2)}\}$ is convergent. Continuing in this way, we construct subsequences $\{v_k^{(j)}\}$ for which $\{K_mv_k^{(j)}\}$ is convergent for all $1 \leq m \leq j$. Next, we let $\{u_j := v_j^{(j)}\}$, the “diagonal” sequence. This is a subsequence of all of the $\{v_k^{(j)}\}$’s. Consequently, for $n$ fixed, $\{K_nu_j\}_{j=1}^{\infty}$ will be convergent. To finish up, we will use an “up, over, and around” argument. Note that for all $\ell, m$,

$$\|Ku_\ell - Ku_m\| \leq \|Ku_\ell - K_nu_\ell\| + \|K_nu_\ell - K_nu_m\| + \|K_nu_m - Ku_m\|$$

Since $\|Ku_\ell - K_nu_\ell\| \leq \|K - K_n\|_{op} \|u_\ell\| \leq 2C\|K - K_n\|_{op}$ and, similarly, $\|K_nu_m - Ku_m\| \leq 2C\|K - K_n\|_{op}$, so $\|Ku_\ell - K_nu_m\| \leq 4C\|K - K_n\|_{op} + \|K_nu_\ell - K_nu_m\|$. Let $\varepsilon > 0$. First choose $N$ such that for $n \geq N$, $\|K - K_n\|_{op} < \varepsilon/(8C)$. Fix $n$. Because $\{K_nu_\ell\}$ is convergent, it is Cauchy. Choose $N'$ so large that $\|K_nu_\ell - K_nu_m\| < \varepsilon/2$ for all $\ell, m \geq N'$. Putting these two together yields $\|Ku_\ell - K_nu_\ell\| \leq \varepsilon$, provided $\ell, m \geq N'$. Thus $\{Ku_\ell\}$ is Cauchy and therefore convergent. \qed

Corollary 2.7. Hilbert-Schmidt operators are compact.

Proof. Let $\mathcal{H} = L^2[0,1]$ and suppose $k(x,y) \in L^2(R)$, $R = [0,1] \times [0,1]$. The associated Hilbert-Schmidt operator is $Ku = \int_0^1 k(x,y)u(y)dy$. Let $\{\phi_n\}_{n=1}^{\infty}$ be an o.n. basis for $L^2[0,1]$. With a little work, one can show that $\{\phi_n(x)\phi_m(y)\}_{n,m=1}^{\infty}$ is an o.n. basis for $L^2(R)$. Also, from Proposition 2 in the notes on \textit{Bounded Operators & Closed Subspaces}, we have that $\|K\|_{op} \leq \|k\|_{L^2(R)}$. Expand $k(x,y)$ in the o.n. basis $\{\phi_n(x)\phi_m(y)\}_{n,m=1}^{\infty}$:

$$k(x,y) = \sum_{n,m=1}^{\infty} \alpha_{m,n}\phi_n(x)\phi_m(y), \quad \alpha_{m,n} = \langle k(x,y), \phi_n(x)\phi_m(y) \rangle_{L^2(R)}$$

\(^2\text{See Keener, Theorem 3.5}\)
Next, let \( k_N(x, y) = \sum_{n,m=1}^{N} \alpha_{m,n} \phi_n(x) \phi_m(y) \) and also \( K_N \) be the finite rank operator \( K_Nu(x) = \int_0^1 k_N(x, y)u(y)dy \). By Parseval’s theorem, we have that

\[
\|k - k_N\|_{L^2(R)}^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\alpha_{m,n}|^2 - \sum_{m=1}^{N} \sum_{n=1}^{N} |\alpha_{m,n}|^2
\]

\[
= \sum_{m=N+1}^{\infty} \sum_{n=1}^{\infty} |\alpha_{m,n}|^2 + \sum_{m=1}^{N} \sum_{n=N+1}^{\infty} |\alpha_{m,n}|^2
\]

\[
\leq \sum_{m=N+1}^{\infty} \sum_{n=1}^{\infty} |\alpha_{m,n}|^2 + \sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} |\alpha_{m,n}|^2.
\]

Both terms go to 0 as \( N \to \infty \). To make this clear, let \( \tilde{a}_m^2 = \sum_{n=1}^{\infty} \alpha_{m,n}^2 \). Because \( \sum_{m=1}^{\infty} \tilde{a}_m^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\alpha_{m,n}|^2 \), the series \( \sum_{m=1}^{\infty} \tilde{a}_m^2 \) is absolutely convergent; consequently, \( \lim_{N \to \infty} \sum_{m=N+1}^{\infty} \tilde{a}_m^2 = 0 \). Using this for both terms in the inequality implies that \( \lim_{N \to \infty} \|k - k_N\|_{L^2(R)}^2 = 0 \). As we mentioned above, \( \|K - K_N\|_{op} \leq \|k - k_N\|_{L^2(R)} \), so

\[
\lim_{N \to \infty} \|K - K_N\|_{op} = 0.
\]

Thus \( K \) is the limit in \( \mathcal{B}(L^2[0, 1]) \) of finite rank operators, which are compact. By Theorem 2.6 above, \( K \) is also compact. \( \square \)

We now turn to some of the algebraic properties of \( \mathcal{C}(\mathcal{H}) \).

**Proposition 2.8.** Let \( K \in \mathcal{C}(\mathcal{H}) \) and let \( L \in \mathcal{B}(\mathcal{H}) \). Then both \( KL \) and \( LK \) are in \( \mathcal{C}(\mathcal{H}) \).

**Proof.** Let \( \{v_k\} \) be a bounded sequence in \( \mathcal{H} \). Since \( L \) is bounded, the sequence \( \{Lv_k\} \) is also bounded. Because \( K \) is compact, we may find a subsequence of \( \{KLv_k\} \) that is convergent, so \( KL \in \mathcal{C}(\mathcal{H}) \). Next, again assuming \( \{v_k\} \) is a bounded sequence in \( \mathcal{H} \), we may extract a convergent subsequence from \( \{Kv_k\} \), which, with a slight abuse of notation, we will denote by \( \{Kv_j\} \). Because \( L \) is bounded, it is also continuous. Thus \( \{LKv_j\} \) is convergent. It follows that \( LK \) is compact. \( \square \)

**Proposition 2.9.** \( K \) is compact if and only if \( K^* \) is compact.

**Proof.** Because \( K \) is compact, it is bounded and so is its adjoint \( K^* \), in fact \( \|K^*\|_{op} = \|K\|_{op} \). By Proposition 2.8, we thus have that \( KK^* \) is compact. It follows that if \( \{u_n\} \) is a bounded sequence in \( \mathcal{H} \), then we may extract a
subsequence of \{u_n\}, denoted by \{v_j\}, such that \(KK^*v_j\) is convergent. This of course means that this sequence is also Cauchy. Note that
\[
\langle KK^*(v_j - v_k), v_j - v_k \rangle = \langle K^*(v_j - v_k), K^*(v_j - v_k) \rangle = \|K^*(v_j - v_k)\|^2.
\]
From this and the fact that \{v_j\}, being a subsequence of the bounded sequence \{u_n\}, is itself bounded, we see that
\[
\|v_j - v_k\| \|KK^*(v_j - v_k)\| \leq C\|KK^*(v_j - v_k)\|.
\]
Thus,
\[
\|K^*(v_j - v_k)\|^2 \leq C\|KK^*(v_j - v_k)\|
\]
Since \(KK^*v_j\) is Cauchy, for every \(\varepsilon > 0\), we can find \(N\) such that whenever \(j, k \geq N\), \(\|KK^*(v_j - v_k)\| < \varepsilon^2/C\). It follows that \(\|K^*(v_j - v_k)\| < \varepsilon\), if \(j, k \geq N\). This implies that \(K^*v_j\) is Cauchy and therefore convergent.

We want to put this in more algebraic language. Taking \(L\) to be compact in Proposition 2.8, we have that the product of two compact operators is compact. Since \(C(H)\) is already a subspace, this implies that it is an algebra. Moreover, by taking \(L\) to be just a bounded operator, we have that \(C(H)\) is a two-sided ideal in the algebra \(B(H)\). Since \(K\) being compact implies \(K^*\) is compact, \(C(H)\) is closed under the operation of taking adjoints; thus, \(C(H)\) is a \(\ast\)-ideal. Finally, by Theorem 2.6, we have that \(C(H)\) is a closed subspace of \(B(H)\). We summarize these results as follows.

**Theorem 2.10.** \(C(H)\) is a closed, two-sided, \(\ast\)-ideal in \(B(H)\).

We remark that a closed \(\ast\)-algebra in \(B(H)\) is called a C*-algebra. So, \(C(H)\) is a C*-algebra that is also a two-sided ideal in \(B(H)\).