Lagrange Multipliers in the Calculus of Variations

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The problem[1] that we wish to address is the following: Consider the functionals $J(y) = \int_a^b f(x, y, y')dx$ and $K(y) = \int_a^b g(x, y, y')dx$. Extremize (maximize/minimize) J over all y such that $y \in C^1[a, b]$, $y(a) = A$, $y(b) = B$, and $K(y) = L$, where L is a constant.

Theorem 0.1. *If $y = \overline{y}$ extremizes $J(y)$, subject to the conditions above, then there exists a constant λ such that $\overline{y}$ is an unconstrained extremal for $D(y) = \int_a^b (f - \lambda g)dx$, $y(a) = A, y(b) = B$.*

Proof. Let $F(x) = \frac{\partial f}{\partial y'}(x, \bar{y}, \bar{y}') - \int_a^x \frac{\partial f}{\partial y}(t, \bar{y}(t), \bar{y}'(t))dt - c_1$, where c_1 is determined by $\int_a^b F(x)dx = 0$. Similarly, let $G(x) = \frac{\partial g}{\partial y'} - \int_a^x \frac{\partial g}{\partial y} dt - c_2$, where $\int_a^b G(x)dx = 0$. (Both F and G are evaluated with $y = \overline{y}$, so they are fixed functions of x.) Next, consider the function

$$\widetilde{y} := \overline{y} + \varepsilon \int_a^x F(t)dt + \delta \int_a^x G(t)dt.$$

This function satisfies the boundary conditions at $x = a$ and $x = b$, because

$$\widetilde{y}(a) = \overline{y}(a) + \epsilon \cdot 0 + \delta \cdot 0 = \overline{y}(a) = A,$$

and

$$\widetilde{y}(b) = \overline{y}(b) + \varepsilon \underbrace{\int_a^b F(x)dx}_{0} + \delta \underbrace{\int_a^b G(x)dx}_{0} = B.$$

We can require it to satisfy the constraint, too. The quantity $K(\widetilde{y}) = \int_a^b g(x, \widetilde{y}, \widetilde{y}')dx$ depends only on the parameters ϵ and δ. The reason is that $\tilde{y}$ depends on known functions, $\overline{y}$, $F(x)$ and $G(x)$. These are integrated out when $K(\widetilde{y})$ is computed. The only variables left are ε and δ. The same is true when $J(\widetilde{y})$ is computed. Thus, we may let $\phi(\varepsilon, \delta) := J(\widetilde{y})$ and $\psi(\varepsilon, \delta) := K(\widetilde{y})$. The constraint is satisfied by requiring $\psi(\varepsilon, \delta) = L$, which implicitly defines a curve relating δ and ε.

[1] The proof presented here may be found in Akhiezer's book on the Calculus of Variations. See the list of references for a full citation.

The extremum for J occurs at $\varepsilon = \delta = 0$, because $\phi(0,0) = J(\overline{y})$, subject to the constraint $\psi(\varepsilon,\delta) = L$. This is equivalent to a 2D Lagrange multiplier problem: Extremize

$$\phi(\varepsilon,\delta) - \lambda(\psi(\varepsilon,\delta) - L),$$

with no constraint. Making use of the extremum occurring at $(0,0)$ gives us

$$\frac{\partial\phi}{\partial\varepsilon}(0,0) = \lambda\frac{\partial\psi}{\partial\varepsilon}(0,0) \text{ and } \frac{\partial\phi}{\partial\delta}(0,0) = \lambda\frac{\partial\psi}{\partial\delta}(0,0). \tag{0.1}$$

We need to compute the various derivatives in the equation above. For $\frac{\partial\phi}{\partial\varepsilon}(0,0)$, we have

$$\frac{\partial\phi}{\partial\varepsilon}(0,0) = \frac{\partial}{\partial\varepsilon}J(\widetilde{y})|_{(0,0)} = \int_a^b \left(\frac{\partial f}{\partial y}\int_a^x F(t)dt + \frac{\partial f}{\partial y'}F(x)\right)dx.$$

Integrating the first term on the right by parts then yields

$$\begin{aligned}\frac{\partial\phi}{\partial\varepsilon}(0,0) &= \int_a^b \Bigg(\underbrace{\frac{\partial f}{\partial y'} - \int_a^x \frac{\partial f}{\partial y}dt}_{F(x)+c_1}\Bigg)F(x)dx \\ &= \int_a^b F(x)^2dx + c_1\underbrace{\int_a^b F(x)dx}_{0} = \langle F,F\rangle\end{aligned}$$

Similar calculations result in

$$\frac{\partial\psi}{\partial\varepsilon}(0,0) = \langle F,G\rangle,\quad \frac{\partial\psi}{\partial\delta}(0,0) = \langle G,G\rangle, \text{ and } \frac{\partial\phi}{\partial\delta}(0,0) = \langle F,G\rangle.$$

Using these in (0.1) and putting the resulting equations in matrix form, we see that

$$\begin{pmatrix}\langle F,F\rangle & \langle F,G\rangle \\ \langle F,G\rangle & \langle G,G\rangle\end{pmatrix}\begin{pmatrix}1 \\ -\lambda\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$

By multiplying on the left by $(1 \quad -\lambda)$ and collecting terms, we arrive at $\|F - \lambda G\|^2 = 0$, which implies that $F - \lambda G = 0$; consequently,

$$\frac{\partial(f-\lambda g)}{\partial y'} - \int_a^x \frac{\partial(f-\lambda g)}{\partial y}dt = c,$$

which is the du Bois-Reymond form of the Euler-Lagrange equations for $D(y)$. These hold for $y = \overline{y}$ and do not involve the constraint, as required. □