Courant-Fischer Theorem for Sturm-Liouville Problems

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1 Variational Characterizations

Let $Lu = -(pu')' + qu$, where p is $C^2[0,1]$, $p(x) > 0$ on $[0,1]$ and let q be $C^2[0,1]$. In addition.we suppose that the domain of L is

$$\mathcal{D} := \{u \in L^2[0,1] \colon Lu \in L^2[0,1],\ u(0) = 0, u'(1) + \sigma u(1) = 0\},$$

with $\sigma > 0$ being a parameter. Under these assumptions, L is self adjoint, and bounded below by some constant c; i.e., $\langle Lu, u\rangle \geq c\|u\|^2$ for all $u \in \mathcal{D}$. By Theorem 4.7 in Keener's book, L has a complete set of orthornormal eigenfunctions $\{\phi_k \in L^2[0,1]\}_{k=1}^{\infty}$ corresponding to eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$. We want to cast finding the eigenfunctions and eigenvalues into the form of a variational problem.

Consider the functional below, where $\mathcal{A} = \{u \in C^2[0,1] : u(0) = 0\}$ is the set of admissible functions:

$$D(u) := \int_0^1 (pu'^2 + qu^2)dx + p(1)\sigma u(1)^2. \tag{1.1}$$

We wish to extremize $D(u)$, subject to the constraint $K(u) := \int_0^1 u^2 dx = 1$. We will use Lagrange multipliers to do this. We will thus need to extremize

$$J(u) = D(u) - \lambda K(u) = \int_0^1 (pu'^2 + qu^2 - \lambda u^2)dx + p(1)\sigma u(1)^2,$$

subject to natural boundary conditions at $x = 1$ and Dirichlet conditions at $x = 0$.

Let u be an extremal. If $z \in C^2[0,1]$ satisfies $z(0) = 0$, and is free at $x = 1$, then $u + \varepsilon z$ is in $\mathcal{A}$. To find u, calculate the Gâteaux derivative and set it equal to 0:

$$\frac{d}{d\varepsilon} J(u + \varepsilon z)|_\varepsilon = 2\int_0^1 (pu'z' + quz - \lambda uz)dx = 0.$$

As usual, integrating by parts and noting that $z(0) = 0$, we see that

$$2\int_0^1 (\underbrace{-pu'' + qu}_{Lu} - \lambda u) z dx + 2p(1)u'(1)z(1) + 2p(1)\sigma u(1)z(1) = 0.$$

Apart from $z(0) = 0$, z is otherwise arbitrary. The usual argument employing the fundamental theorem of the calculus of variations implies that

$$Lu - \lambda u = 0. \tag{1.2}$$

Using this in the previous equation and dividing by 2p(1), we obtain $(u'(1) + \sigma u(1))z(1) = 0$. Because $z(1)$ is arbitrary, we may set it equal to 1. The end result is that

$$u'(1) + \sigma u(1) = 0. \tag{1.3}$$

By (1.3), (1.2), u being in $\mathcal{A}$ and $Lu \in L^2[0,1]$, we have that u is in $\mathcal{D}$, the domain of L. Thus u is an eigenfunction of L corresponding to λ.

2 The Courant-Fischer Theorem

Suppose that $u \in \mathcal{D}$. Integrating $\langle Lu, u\rangle = \int_0^1 (-(pu')' + qu)u dx$ by parts, we obtain

$$\langle Lu, u\rangle = D(u), \tag{2.1}$$

where $D(u)$ is defined in (1.1). Because the eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ form a complete orthonormal set in $L^2[0,1]$, we may expand both Lu and u in them. To begin, we have that

$$Lu = \sum_{k=1}^{\infty} \langle Lu, \phi_k\rangle \phi_k.$$

Since L is self adjoint, $\langle Lu, \phi_k\rangle = \langle u, L\phi_k\rangle = \lambda_k \langle u, \phi_k\rangle$. Since u is also in $L^2[0,1]$, we can expand it in the ϕ_k's: $u = \sum_{k=1}^{\infty} \langle u, \phi_k\rangle \phi_k$. From these two expansions and (2.1), we see that[1]

$$D(u) = \langle Lu, u\rangle = \sum_{k=1}^{\infty} \lambda_k \underbrace{\langle u, \phi_k\rangle}_{\alpha_k}{}^2 = \sum_{k=1}^{\infty} \lambda_k \alpha_k^2. \tag{2.2}$$

[1] We are working in a real Hilbert space.

Theorem 2.1 (Courant-Fischer)**.** *Let $S_{k-1} = \{\psi_1, \psi_2, \ldots, \psi_{k-1}\}$ be a set of arbitrary vectors in $L^2[0,1]$ and suppose that $\|u\|^2 = \sum_{k=1}^{\infty} \alpha_k^2 = 1$, then*

$$\begin{cases} \lambda_k = \max\limits_{S_{k-1}} \min\limits_{u \in S_{k-1}^{\perp},\, \|u\|=1} D(u) \\ \lambda_1 = \min\limits_{\|u\|=1} D(u) \end{cases}$$

Proof. Suppose that $k \geq 2$ and $\psi_j = \phi_j$, $j = 1, \ldots, k-1$. Then, $\alpha_j = 0$ for $j = 1$ to $k-1$. Since $\alpha_j = \langle u, \phi_j \rangle$, we have $\alpha_j = 0$ for $j \leq k-1$. Thus, $u = \sum_{n=k} \alpha_n \phi_n$ and $\|u\|^2 = \sum_{n=k}^{\infty} \alpha_n^2 = 1$. It follows that

$$D(u) = \sum_{n=k}^{\infty} \lambda_n \alpha_n^2 \geq \lambda_k \sum_{n=k}^{\infty} \alpha_n^2 = \lambda_k \cdot 1 = \lambda_k = D(\phi_k)$$

One thus has that $\lambda_k = D(\phi_k) \leq \min_{u \in S_{k-1}^{\perp},\, \|u\|=1} D(u)$ for $\psi_j = \phi_j$, $j = 1, \ldots, k-1$.

We now need to show that for an arbitrary S_{k-1}, $\min_{u \in S_{k-1}^{\perp},\, \|u\|=1} D(u) \leq D(\phi_k)$. Expand each ψ_j in the ϕ_k basis.

$$\psi_j = \sum_{n=1}^{\infty} \underbrace{\langle \psi_j, \phi_n \rangle}_{\beta_{j,n}} \phi_n = \sum_{n=1}^{\infty} \beta_{j,n} \phi_n$$

Note that if we choose $\alpha_n = 0$ for $n \geq k+1$, but otherwise arbitrary, we have

$$0 = \langle u, \psi_j \rangle = \sum_{n=1}^{\infty} \beta_{j,n} \alpha_n = \sum_{n=1}^{k} \beta_{j,n} \alpha_n.$$

The number of equations above is $k-1$, one for each j, and the number of variables is k. Since the system is homogeneous, we have at least one nonzero solution, which we can normalize so that $\sum_{n=1}^{k} \alpha_n^2 = 1$. This implies that $u = \sum_{n=1}^{k} \alpha_n \phi_n$ is in $S_{k-1}^{\perp}$ and has $\|u\| = 1$, so it satisfies the required conditions. Next, we have

$$D(u) = \sum_{n=1}^{k} \lambda_n \alpha_n^2 \leq \lambda_k \sum_{n=1}^{k} \alpha_n^2 = \lambda_k \cdot 1 = \lambda_k.$$

Since this holds for one u satisfying the required conditions, the u that minimizes $D(u)$ subject to these conditions has to be smaller than the value of

$D(u)$ for the particular u we found above. Thus, $\min_{u \in S_{k-1}^{\perp}, \|u\|=1} D(u) \leq \lambda_k = D(\phi_k)$. Consequently, we obtain, $\lambda_k = \max_{S_{k-1}} \min_{u \in S_{k-1}^{\perp}, \|u\|=1} D(u)$, for $k \geq 2$. The $k = 1$ case can be proven via a similar argument. $\square$