Courant-Fischer Theorem for Sturm-Liouville Problems
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1 Variational Characterizations

Let $Lu = -(pu')' + qu$, where $p$ is $C^2[0,1]$, $p(x) > 0$ on $[0,1]$ and let $q$ be $C^2[0,1]$. In addition, we suppose that the domain of $L$ is

$$D := \{ u \in L^2[0,1] : Lu \in L^2[0,1], \ u(0) = 0, u'(1) + \sigma u(1) = 0 \},$$

with $\sigma > 0$ being a parameter. Under these assumptions, $L$ is self adjoint, and bounded below by some constant $c$; i.e., $\langle Lu, u \rangle \geq c \| u \|^2$ for all $u \in D$. By Theorem 4.7 in Keener’s book, $L$ has a complete set of orthonormal eigenfunctions $\{ \phi_k \in L^2[0,1] \}_{k=1}^{\infty}$ corresponding to eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$. We want to cast finding the eigenfunctions and eigenvalues into the form of a variational problem.

Consider the functional below, where $A = \{ u \in C^2[0,1] : u(0) = 0 \}$ is the set of admissible functions:

$$D(u) := \int_0^1 (pu'^2 + qu^2)dx + p(1)\sigma u(1)^2. \quad (1.1)$$

We wish to extremize $D(u)$, subject to the constraint $K(u) := \int_0^1 u^2 dx = 1$. We will use Lagrange multipliers to do this. We will thus need to extremize

$$J(u) = D(u) - \lambda K(u) = \int_0^1 (pu'^2 + qu^2 - \lambda u^2)dx + p(1)\sigma u(1)^2,$$

subject to natural boundary conditions at $x = 1$ and Dirichlet conditions at $x = 0$.

Let $u$ be an extremal. If $z \in C^2[0,1]$ satisfies $z(0) = 0$, and is free at $x = 1$, then $u + \varepsilon z$ is in $A$. To find $u$, calculate the Gâteaux derivative and set it equal to 0:

$$\frac{d}{d\varepsilon} J(u + \varepsilon z)|_{\varepsilon} = 2 \int_0^1 (pu'z' + quz - \lambda uz)dx = 0.$$
As usual, integrating by parts and noting that \( z(0) = 0 \), we see that

\[
2 \int_0^1 \frac{(-pu'' + qu - \lambda u)z}{Lu} dx + 2p(1)u'(1)z(1) + 2p(1)\sigma u(1)z(1) = 0.
\]

Apart from \( z(0) = 0 \), \( z \) is otherwise arbitrary. The usual argument employing the fundamental theorem of the calculus of variations implies that

\[
Lu - \lambda u = 0. \tag{1.2}
\]

Using this in the previous equation and dividing by \( 2p(1) \), we obtain \( (u'(1) + \sigma u(1))z(1) = 0 \). Because \( z(1) \) is arbitrary, we may set it equal to 1. The end result is that

\[
u'(1) + \sigma u(1) = 0. \tag{1.3}
\]

By (1.3) and (1.2), \( u \) being in \( A \) and \( Lu \in L^2[0,1] \), we have that \( u \) is in \( D \), the domain of \( L \). Thus \( u \) is an eigenfunction of \( L \) corresponding to \( \lambda \).

2 The Courant-Fischer Theorem

Suppose that \( u \in D \). Integrating \( \langle Lu, u \rangle = \int_0^1 (-pu'') + quu \, dx \) by parts, we obtain

\[
\langle Lu, u \rangle = D(u), \tag{2.1}
\]

where \( D(u) \) is defined in (1.1). Because the eigenfunctions \( \{ \phi_k \}_{k=1}^{\infty} \) form a complete orthonormal set in \( L^2[0,1] \), we may expand both \( Lu \) and \( u \) in them.

To begin, we have that

\[
Lu = \sum_{k=1}^{\infty} \langle Lu, \phi_k \rangle \phi_k.
\]

Since \( L \) is self adjoint, \( \langle Lu, \phi_k \rangle = \langle u, L\phi_k \rangle = \lambda_k \langle u, \phi_k \rangle \). Since \( u \) is also in \( L^2[0,1] \), we can expand it in the \( \phi_k \)'s: \( u = \sum_{k=1}^{\infty} \langle u, \phi_k \rangle \phi_k \). From these two expansions and (2.1), we see that

\[
D(u) = \langle Lu, u \rangle = \sum_{k=1}^{\infty} \lambda_k \langle u, \phi_k \rangle^2 = \sum_{k=1}^{\infty} \lambda_k \alpha_k^2. \tag{2.2}
\]

\(^1\)We are working in a real Hilbert space.
Theorem 2.1 (Courant-Fischer). Let $S_{k-1} = \{\psi_1, \psi_2, \ldots, \psi_{k-1}\}$ be a set of arbitrary vectors in $L^2[0,1]$ and suppose that $\|u\|^2 = \sum_{k=1}^{\infty} \alpha_k^2 = 1$, then

$$
\begin{align*}
\lambda_k &= \max_{S_{k-1}} \min_{u \in S_{k-1}, \|u\|=1} D(u) \\
\lambda_1 &= \min_{\|u\|=1} D(u)
\end{align*}
$$

Proof. Suppose that $k \geq 2$ and $\psi_j = \phi_j$, $j = 1, \ldots, k-1$. Then, $\alpha_j = 0$ for $j = 1$ to $k-1$. Since $\alpha_j = \langle u, \phi_j \rangle$, we have $\alpha_j = 0$ for $j \leq k-1$. Thus, $u = \sum_{n=k}^{\infty} \alpha_n \phi_n$ and $\|u\|^2 = \sum_{n=k}^{\infty} \alpha_n^2 = 1$. It follows that

$$
D(u) = \sum_{n=k}^{\infty} \lambda_n \alpha_n^2 \geq \lambda_k \sum_{n=k}^{\infty} \alpha_n^2 = \lambda_k \cdot 1 = \lambda_k = D(\phi_k)
$$

One thus has that $\lambda_k = D(\phi_k) \leq \min_{u \in S_{k-1}, \|u\|=1} D(u)$ for $\psi_j = \phi_j$, $j = 1, \ldots, k-1$.

We now need to show that for an arbitrary $S_{k-1}$, $\min_{u \in S_{k-1}, \|u\|=1} D(u) \leq D(\phi_k)$. Expand each $\psi_j$ in the $\phi_k$ basis.

$$
\psi_j = \sum_{n=1}^{\infty} \langle \psi_j, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \beta_{j,n} \phi_n
$$

Note that if we choose $\alpha_n = 0$ for $n \geq k+1$, but otherwise arbitrary, we have

$$
0 = \langle u, \psi_j \rangle = \sum_{n=1}^{\infty} \beta_{j,n} \alpha_n = \sum_{n=1}^{k} \beta_{j,n} \alpha_n.
$$

The number of equations above is $k-1$, one for each $j$, and the number of variables is $k$. Since the system is homogeneous, we have at least one nonzero solution, which we can normalize so that $\sum_{n=1}^{k} \alpha_n^2 = 1$. This implies that $u = \sum_{n=1}^{k} \alpha_n \phi_n$ is in $S_{k-1}$ and has $\|u\| = 1$, so it satisfies the required conditions. Next, we have

$$
D(u) = \sum_{n=1}^{k} \lambda_n \alpha_n^2 \leq \lambda_k \sum_{n=1}^{k} \alpha_n^2 = \lambda_k \cdot 1 = \lambda_k.
$$

Since this holds for one $u$ satisfying the required conditions, the $u$ that minimizes $D(u)$ subject to these conditions has to be smaller than the value of
$D(u)$ for the particular $u$ we found above. Thus, $\min_{u \in S_{k-1}^+} \|u\| = 1 D(u) \leq \lambda_k = D(\phi_k)$. Consequently, we obtain, $\lambda_k = \max_{S_{k-1}} \min_{u \in S_{k-1}^+} \|u\| = 1 D(u)$, for $k \geq 2$. The $k = 1$ case can be proven via a similar argument. \hfill \Box$