Graphs and Operators
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August, 2015

1 The graph of an operator

Let $\mathcal{H}$ be a separable Hilbert space. If $L : \mathcal{H} \to \mathcal{H}$ is a linear operator defined on a domain $D_L \subseteq \mathcal{H}$, then the *graph* of L is defined by

$$G_L := \{(u, Lu) \in \mathcal{H} \times \mathcal{H} \colon u \in D_L\}.$$

We can turn $\mathcal{H} \times \mathcal{H}$ into a vector space by defining addition and multiplication by a scalar via $(u_1, u_2) + (v_1, v_2) := (u_1 + u_2, v_1 + v_2)$ and $c(u_1, u_2) := (cu_, cu_2)$. In addition, we make $\mathcal{H} \times \mathcal{H}$ into an inner product space $\mathbf{H}$ by defining the inner product this way:

$$\langle (u_1, u_2), (v_1, v_2)\rangle_{\mathbf{H}} = \langle u_1, v_1\rangle_{\mathcal{H}} + \langle u_2, v_2\rangle_{\mathcal{H}}.$$

It is very easy to show that $\mathbf{H}$ is a Hilbert space. We leave the verification to the reader.

It's notationally convenient to represent a vector in $\mathbf{H}$ as a column

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The vector operations in $\mathbf{H}$ are equivalent to those for column vectors, except of course that the entries in the column are in $\mathcal{H}$. In this notation, we have

$$G_L = \{\mathbf{u} \in \mathbf{H} \colon \mathbf{u} = \begin{pmatrix} u & Lu \end{pmatrix}^T\}.$$

This set is algebraically closed under vector addition and scalar multiplication; it is thus a subspace of $\mathbf{H}$.

Bounded operators are continuous. While unbounded operators cannot be continuous, for some unbounded operators there is an important property that plays a role similar to continuity in a bounded operator:

Definition 1.1. *An operator L is said to be closed if and only if for every sequence $\{u_k\}_{k=1}^{\infty}$ such that both $u_k \to u$ and $Lu_k \to v$, in $\mathcal{H}$, we have $u \in D_L$ and $v = Lu$.*

We can rephrase the definition of a closed operator in terms of its graph. Recall that a subspace of a Hilbert space is closed if and only if every convergent sequence in the subspace has a limit in that subspace. In particular, G_L will be closed if and only if every sequence $\{\mathbf{u}_k\} \subset G_L$ that converges in $\mathbf{H}$ has its limit in G_L. This means that G_L is closed if and only if $u_k \to u$ and $Lu_k \to v$, in $\mathcal{H}$, then $u \in D_L$ and $v = Lu$. We have thus proved the following proposition.

Proposition 1.2. *The graph G_L is a closed subspace of $\mathbf{H}$ if and only if the operator L is closed.*

2 Adjoints

2.1 Definition

We start with D^*, which will be the domain of the adjoint of L. Define D^* to be the set of all $v \in \mathcal{H}$ such that there exists $\tilde{v} \in \mathcal{H}$ such that

$$\langle Lu, v\rangle_{\mathcal{H}} = \langle u, \tilde{v}\rangle_{\mathcal{H}}, \text{ for all } u \in D_L. \tag{2.1}$$

The set D^* is also a subspace of $\mathcal{H}$. Suppose that v_1 and v_2 are in D^*. Then we have

$$\begin{aligned}\langle Lu, c_1v_1 + c_2v_2\rangle_{\mathcal{H}} &= \bar{c}_1\langle u, \tilde{v}_1 + \bar{c}_2\langle u, \tilde{v}_2\rangle \\ &= \langle u, c_1\tilde{v}_1 + c_2\tilde{v}_2\rangle_{\mathcal{H}},\end{aligned}$$

so $c_1v_1 + c_2v_2 \in D^*$. Hence, D^* is algebraically closed under addition and scalar multiplication and is thus a subspace of $\mathcal{H}$.

We now turn to a discussion of the adjoint, L^*. Suppose D_L is dense in $\mathcal{H}$. Then, if $v \in D^*$, we have that, for all $u \in D_L$, $\langle Lu, v\rangle_{\mathcal{H}} = \langle u, \tilde{v}\rangle_{\mathcal{H}}$. Suppose that there are two vectors $\tilde{v}_1$ and $\tilde{v}_2$ such that $\langle u, \tilde{v}_1\rangle_{\mathcal{H}} = \langle u, \tilde{v}_2\rangle_{\mathcal{H}}$. Subtracting, we see that $\langle u, \tilde{v}_1 - \tilde{v}_2\rangle_{\mathcal{H}} = 0$. Since D_L is dense in $\mathcal{H}$, taking limits in the previous equation shows that $\langle u, \tilde{v}_1 - \tilde{v}_2\rangle_{\mathcal{H}} = 0$ for *all* $u \in \mathcal{H}$. Picking $u = \tilde{v}_1 - \tilde{v}_2$ then implies that $\|\tilde{v}_1 - \tilde{v}_2\|_{\mathcal{H}} = 0$ and that $\tilde{v}_1 = \tilde{v}_2$. Consequently, $\tilde{v}$ is uniquely determined by v and, thus, we may define a map $L^* : D^* \to \mathcal{H}$ via $L^*v = \tilde{v}$.

The map L^* is linear. To prove this, recall that D^* is a subspace of $\mathcal{H}$. It follows that, for $v_1, v_2 \in D^*$, we have that

$$\langle Lu, c_1v_1 + c_2v_2\rangle_{\mathcal{H}} = \langle u, L^*(c_1v_1 + c_2v_2)\rangle_{\mathcal{H}}$$

and, in addition, that

$$\langle Lu, c_1v_1 + c_2v_2\rangle_{\mathcal{H}} = \bar{c}_1\langle Lu, v_1\rangle_{\mathcal{H}} + c_2\langle Lu, v_2\rangle_{\mathcal{H}} = \langle u, c_1L^*v_1 + c_2L^*v_2\rangle_{\mathcal{H}}.$$

Consequently, we see that

$$L^*(c_1v_1 + c_2v_2) = c_1L^*v_1 + c_2L^*v_2.$$

It follows that L^* is a linear operator having domain $D_{L^*} = D^*$. We summarize these remarks in the following proposition.

Theorem 2.1. *Let L be densely defined and let D^* be as above. Then there exists a linear operator L^*, called the adjoint of L, with domain $D_{L^*} = D^*$, for which $\langle Lu, v\rangle_{\mathcal{H}} = \langle u, L^*v\rangle_{\mathcal{H}}$ holds for all $u \in D_L$ and all $v \in D_{L^*}$.*

2.2 Graphs and properties of adjoints

There is an important relation between the graphs G_L and G_{L^*}. We begin by noting that there is another way of characterizing D_{L^*}. By definition, a vector $v \in D_{L^*}$ if and only if there is a $\tilde{v} \in \mathcal{H}$ such that $\langle Lu, v\rangle_{\mathcal{H}} = \langle u, \tilde{v}\rangle_{\mathcal{H}}$ holds. We can put this in terms of $\mathbf{H}$:

$$\left\langle U \begin{pmatrix} u \\ Lu \end{pmatrix}, \begin{pmatrix} v \\ \tilde{v} \end{pmatrix} \right\rangle_{\mathbf{H}} = \langle Lu, v\rangle_{\mathcal{H}} - \langle u, \tilde{v}\rangle_{\mathcal{H}} = 0,\ U := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.2}$$

Conversely, if (2.2) holds, we have that $\langle Lu, v\rangle_{\mathcal{H}} = \langle u, \tilde{v}\rangle_{\mathcal{H}}$. Since there are no other conditions on v and $\tilde{v}$, it follows that all $\begin{pmatrix} v & \tilde{v} \end{pmatrix}^T$ satisfying the previous equation comprise the orthogonal complement of the space UG_L, $[UG_L]^\perp$. Furthermore, we have that $\tilde{v} = L^*v$, $v \in D_{L^*}$; thus these vectors have the form $\begin{pmatrix} v & L^*v \end{pmatrix}^T$ and so comprise G_{L^*}. This yields the following result[1]

Proposition 2.2. *If L is a densely defined operator, then the following hold: $G_{L^*} = [UG_L]^\perp$, G_{L^*} is closed, and L^* is a closed operator.*

Proof. We have already proved the first assertion. The set UG_L is of course a subspace of $\mathbf{H}$. Hence, because an orthogonal complement of a subspace of a Hilbert space is always closed, G_{L^*} is closed. That L^* is closed follows from Proposition 1.2. □

So far, we have assumed only that L is densely defined. What happens if L is also closed? Here is the answer.

Proposition 2.3. *If L is a densely defined, closed operator, then L^* is a densely defined, closed operator.*

[1] In class I gave another proof, which didn't use graphs, that L being densely defined implies that L^* is closed.

Proof. We have already shown in Proposition 2.2 that if L is densely defined, then L^* is closed.

If L is closed, then, by Proposition 1.2, the graph G_L is closed, and so is UG_L. Recall that if V and W are closed subspaces of a Hilbert space, and if $V = W^\perp$, then we also have $W = V^\perp$. It follows that $G_{L^*} = [UG_L]^\perp$ implies that $G_{L^*}^\perp = UG_L$. Now, suppose that D_{L^*} is *not* dense in $\mathcal{H}$. Then we can find $h \in \mathcal{H}$, $h \neq 0$, such that h is orthogonal to D_{L^*}. Hence, we have that

$$\left\langle \begin{pmatrix} h \\ 0 \end{pmatrix}, \begin{pmatrix} v \\ L^*v \end{pmatrix} \right\rangle_{\mathbf{H}} = \langle h, v \rangle_{\mathcal{H}} = 0,$$

so $\begin{pmatrix} h & 0 \end{pmatrix}^T \in UG_L$. Since this space is closed, we know that there exists a $u \in D_L$ such that $\begin{pmatrix} Lu & -u \end{pmatrix}^T = \begin{pmatrix} h & 0 \end{pmatrix}^T$. Consequently, we see that $u = 0$ and so $h = L0 = 0$. But this contradicts the fact that $h \neq 0$; thus, D_{L^*} is dense is $\mathcal{H}$ and L^* is densely defined. □

3 Closed Graph Theorem

We are just going to state the closed graph theoem. For a proof, see [1].

Theorem 3.1 (Closed Graph Theorem)**.** *Let L be a closed linear operator whose domain $D_L = \mathcal{H}$. Then L is bounded.*

References

[1] A. G. Ramm, *A simple proof of the closed graph theorem*, arXiv:1601.02600v1 [mathFA] 9 Jan 2016.