

Math 642
Apr. 17, 2020

(1)

Last time: Spectral families, spectral measures, and the spectral theorem.

Today: Finding the spectral measure.

1. More on the spectral theorem. If $\phi(\lambda)$ is a bounded continuous function, then we can define the operator $\phi(L)$ by

$$\phi(L) = \int_{-\infty}^{\infty} \phi(\lambda) dE_\lambda.$$

In particular, if $\phi(\lambda) = (\lambda - z)^{-1}$, then $\phi(L) = (L - z)^{-1}$, $z \notin \sigma(L)$, then

$$(L - z)^{-1} = \int_{-\infty}^{\infty} (z - \lambda)^{-1} dE_\lambda.$$

2. The Stieltjes inversion formula. Let $\alpha(x)$ be a bounded non-decreasing function defined on \mathbb{R} , then, for $x_0 < x_1$,

$$\frac{1}{2} (\alpha(x_1) + \alpha(x_0^-)) - \frac{1}{2} (\alpha(x_0) + \alpha(x_1^-))$$

$$= \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{x_0}^{x_1} \left(\int_{-\infty}^{\infty} \frac{e^{itx}}{(t-x)^2 + \epsilon^2} d\alpha(t) \right) dx.$$

We won't give a proof here. However, will give an indication as to why it's true.

(2)

Proof. Let $I(x_0, x_1, \varepsilon) = \frac{1}{\pi} \int_{x_0}^{x_1} \int_0^\infty \frac{\varepsilon}{(x-t)^2 + \varepsilon^2} dt$

Proof: Let $I(x_0, x_1, \varepsilon) = \frac{1}{\pi} \int_{x_0}^{x_1} \int_0^\infty \frac{\varepsilon}{(x-t)^2 + \varepsilon^2} dt$,

By Fubini's theorem, and the fact that

$$\int_{x_0}^{x_1} \frac{dt}{(x-t)^2 + \varepsilon^2} = \tan^{-1}\left(\frac{x_1-t}{\varepsilon}\right) - \tan^{-1}\left(\frac{x_0-t}{\varepsilon}\right),$$

we have that

$$I(x_0, x_1, t) = \frac{1}{\pi} \int_{-\infty}^0 \left(\tan^{-1}\left(\frac{x_1-t}{\varepsilon}\right) - \tan^{-1}\left(\frac{x_0-t}{\varepsilon}\right) \right) d\alpha(t).$$

Suppose that $\alpha(t)$ has jumps at x_0 and x_1 . Then the integral divides into five pieces

$$\begin{aligned} I(x_0, x_1, \varepsilon) &= \frac{1}{\pi} \int_{-\infty}^{x_0^-} (\dots) d\alpha(t) + \frac{1}{\pi} (\alpha(x_0) - \alpha(x_0^-)) \left[\tan^{-1}\left(\frac{x_1-x_0}{\varepsilon}\right) \right] \\ &\quad + \frac{1}{\pi} \int_{x_0^+}^{x_1^-} (\dots) d\alpha(t) + \frac{1}{\pi} (\alpha(x_1) - \alpha(x_1^+)) \left[-\tan^{-1}\left(\frac{x_1-x_0}{\varepsilon}\right) \right] \\ &\quad + \frac{1}{\pi} \int_{x_1^+}^\infty (\dots) d\alpha(t). \end{aligned}$$

Observe that $\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} (\tan^{-1}(y/\varepsilon)) = \begin{cases} \frac{1}{2}, & y > 0 \\ 0, & y = 0 \\ -\frac{1}{2}, & y < 0. \end{cases}$

Using this above, along with the assumption that we can interchange limit and integrals we have:

$$\lim_{\varepsilon \downarrow 0} I(x_0, x_1, \varepsilon) = 0 + \frac{1}{2} (\alpha(x_1) - \alpha(x_0^-)) + \int_{x_0}^{x_1^-} d\alpha(t) + \frac{1}{2} (\alpha(x_1^+) - \alpha(x_0))$$

(3)

$$\lim_{\varepsilon \downarrow 0} I(x_0, x_1, \varepsilon)$$

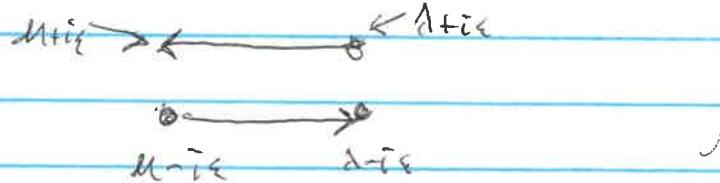
$$= 0 + \frac{1}{2}(\alpha(x_0) - \alpha(x_1^-)) + \int_{x_0}^{x_1^-} d\alpha(t) \\ + \frac{1}{2}(\alpha(x_1) - \alpha(x_1^-)) + 0 \\ = \alpha(x_1^-) - \alpha(x_0)$$

$$= \frac{1}{2}(\alpha(x_1) + \alpha(x_1^-)) - \frac{1}{2}(\alpha(x_0) + \alpha(x_0^-)),$$

which proves the result.
Now,

3. Recovering the spectral measure.

Define Γ_ε to be the "contour" shown,



Then,

$$(E_{\lambda+i\varepsilon} - E_{\lambda-i\varepsilon})_{\text{left}}$$

$$\frac{1}{2}(E_{\lambda+i\varepsilon} + E_{\lambda-i\varepsilon}) - \frac{1}{2}(E_{\mu+i\varepsilon} + E_{\mu-i\varepsilon})$$

$$= \lim_{\varepsilon \downarrow 0} \left(\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (L - z)^{-1} dz \right).$$

(4)

Proof: First, we have that

$$-\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (L-s)^{-1} ds = -\frac{1}{2\pi i} \int_{\Gamma}^1 (L-s-i\varepsilon)^{-1} ds$$

~~($x = s - i\varepsilon$)~~

On bottom, since $s(x) = x - i\varepsilon$,

On top, ~~$s(x) = x + i\varepsilon$~~ ,

$$\begin{aligned} &= -\frac{1}{2\pi i} \int_{\Gamma}^1 ((L-x)^2 + \varepsilon^2)^{-1} (-2i\varepsilon) dx \\ &\stackrel{(1)}{=} \frac{\varepsilon}{\pi} \int_{\Gamma}^1 ((L-x)^2 + \varepsilon^2)^{-1} dx, \end{aligned}$$

Apply the spectral measure theorem

$$\begin{aligned} -\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (L-s)^{-1} ds &= \frac{\varepsilon}{\pi} \int_{\Gamma}^1 \int_{-\infty}^{\infty} ((t-x)^2 + \varepsilon^2)^{-1} dE_t dx \\ &\stackrel{(2)}{=} -\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (L-s)^{-1} ds = \frac{1}{\pi} \int_{\Gamma}^1 \left(\int_{-\infty}^{\infty} \frac{dx}{(x-t)^2 + \varepsilon^2} \right) dE_t. \end{aligned}$$

Next, work with inner products:

$$\begin{aligned} &\left\langle \left(-\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (L-s)^{-1} ds \right) f, g \right\rangle \\ &= \frac{1}{\pi} \int_{\Gamma}^1 \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} \frac{dx}{(x-t)^2 + \varepsilon^2} \right)^{-1}}_{d\alpha(t)} \underbrace{\langle f, g \rangle}_{d\alpha(t)}, \end{aligned}$$

(5)

Let's apply the Shetjier inversion formula to

$$d\alpha(t) = -d \langle E_\lambda f, f \rangle = d\alpha(t)_K.$$

We then have

$$\frac{\alpha(\lambda) + \alpha(\lambda^-)}{2} - \frac{\alpha(\mu) + \alpha(\mu^-)}{2} =$$

$$\lim_{\epsilon \rightarrow 0} \left\langle \left(-\frac{1}{2\pi i} \int_{\gamma_\epsilon^R} (L-s)^{-1} ds \right) f, f \right\rangle.$$

In terms of resolvents — with the limits being strong —, this is equivalent to

$$\begin{aligned} & \cancel{\text{s-limit}} \quad \lim_{\epsilon \rightarrow 0} \left(-\frac{1}{2\pi i} \int_{\mu}^{\lambda} (L-s-i\epsilon)^{-1} ds \right. \\ & \quad \left. + \frac{1}{2\pi i} \int_{\mu}^{\lambda} (L-s+i\epsilon)^{-1} ds \right) \\ &= \frac{1}{2} (E_\lambda + E_{\lambda^-}) - \frac{1}{2} (E_\mu + E_{\mu^-}), \end{aligned}$$

Kernel Form If $G(x, y, t)$ is the Green's function for $(L-2I)$, then we can ~~read~~ write this as

$$\underbrace{\frac{1}{2}(E_\lambda + E_{\lambda^-}) - \frac{1}{2}(E_\mu + E_{\mu^-})}_{\text{Kernel}} = \lim_{\epsilon \rightarrow 0} \left[-\frac{1}{2\pi i} \int_{\mu}^{\lambda} \left\{ G(x, y, s+i\epsilon) - G(x, y, s-i\epsilon) \right\} ds \right]$$

Next time: Fourier transform.

Last time: Finding the spectral measure.

Today: Fourier transforms.

1. The spectral measure in terms of Green's Functions.

Recall that if $\mu < \lambda$, then we have that sign was

$$\frac{1}{2}(E_\lambda + E_{\lambda^-}) - \frac{1}{2}(E_\mu + E_{\mu^-}) = \lim_{\epsilon \downarrow 0} \left[-\frac{1}{2\pi i} \int_0^1 [G(x, y, t+i\epsilon) - G(x, y, t-i\epsilon)] dt \right].$$

This is sometimes called Stone's formula. The spectral measure is given in terms of a kernel, which come from the one for $(L-\lambda I)^{-1}$. This is what $G(x, y, z)$ is. Finally, if the kernel is "nice", we can bring the limit $\mu \rightarrow \lambda$ in the integral. If we do that, we get

$$\frac{1}{2}(E_\lambda + E_{\lambda^-}) - \frac{1}{2}(E_\mu + E_{\mu^-}) =$$

~~$$+ \frac{1}{2\pi i} \int_0^1 (G(x, y, t) - G(x, y, t)) dt,$$~~

$$= \frac{1}{2\pi i} \int_0^1 \underbrace{\left(G(x, y, t^*) - G(x, y, t^*) \right)}_{\text{not } t^* \in \lambda} dt,$$

(2)

2. Green's Function for $L-\lambda f$

$$Lu = -u'', \quad D(L) = \{u \in L^2 : u'' \in L^2\}.$$

$$\text{Comments: } L = L^* \Rightarrow \sigma(L) \subseteq \mathbb{R}.$$

~~Example~~ can show that $\langle Lu, u \rangle \geq 0$ & $u \in D(L)$.

This implies that $\sigma(L) \subseteq [c, \infty)$. In fact,
 $\sigma(L) = \sigma_c(L) = [c, \infty)$.

Green's Funct-

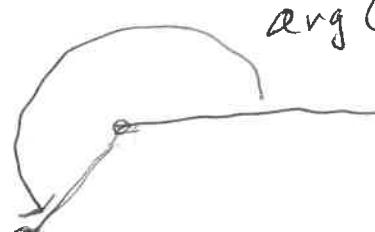
$$\lambda \in \rho(L), \quad G''(x, y, \lambda) - \lambda G = \delta(x-y)$$

$G(x, y, \lambda)$ is in $L^2(\mathbb{R})$ ~~and~~ in $L^1(\mathbb{R})$

Homogeneous solutions to $\pm i\sqrt{\lambda}(x-y)$

choice of Γ : since we want to avoid $\sigma(L)$, we first choose a

$$0 < \arg(\lambda) < 2\pi.$$



we then choose $\bar{\Gamma}$ with a branch cut along the nonnegative real axis:

$$|\bar{\Gamma}| \leq |\lambda|^{1/2} e^{\frac{i}{2}\arg(\lambda)}$$

Note that with this choice, $\operatorname{Im}(\bar{\lambda}) > 0$, because the $\bar{\Gamma}$ is in the upper half plane.

(3)

* Choice of sign

For $x > y$, $G(x, y, \lambda) \in L^1([y, \infty])$ and

For $x < y$, $G(x, y, \lambda) \in L^1(-\infty, y]$
~~we have~~ $\{ e^{\pm i\sqrt{\lambda}(x-y)} \} = e^{\pm i\text{Re}(i\sqrt{\lambda})(x-y)}$

~~we have~~ $\text{Re}(i\sqrt{\lambda}) = -\frac{\text{Im}(\sqrt{\lambda})}{2}$ \leftarrow $\text{Im}(\sqrt{\lambda}) < 0$

~~so~~ $\Rightarrow \text{Re}(i\sqrt{\lambda}(x-y)) = -\text{Im}(\sqrt{\lambda})(x-y)$.

For $x < y$, use +. For $x < y$, $x-y < 0$,

choose "-".

$$G(x, y, \lambda) = \begin{cases} Ae^{-i\sqrt{\lambda}(x-y)} & x < y \\ Be^{+i\sqrt{\lambda}(x-y)} & x > y \end{cases}$$

$$\Rightarrow G(x, y, \lambda) = \begin{cases} A e^{+i\sqrt{\lambda}(y-x)}, & x < y \\ B e^{i\sqrt{\lambda}(x-y)}, & x > y. \end{cases}$$

Continuity. $\frac{B(0^+, y, \lambda)}{B(0^-, y, \lambda)} = G(0, y, \lambda) = A$

$$A = B = \frac{i}{2\sqrt{\lambda}}$$

Jump $G'(0^+, y, \lambda) - G'(0^-, y, \lambda) = -1.$

$$\Rightarrow B(+i\sqrt{\lambda}) - (-i\sqrt{\lambda})/A = -1 \Rightarrow A = B = \frac{-1}{i\sqrt{\lambda}}$$

[4]

$$\Rightarrow G(x, y, \lambda) = \frac{i}{2\sqrt{\lambda}} \begin{cases} e^{-i\sqrt{\lambda}(x-y)}, & x < y \\ e^{i\sqrt{\lambda}(x-y)}, & x > y, \end{cases}$$

$$\Rightarrow G(x, y, \lambda) = \frac{i}{2\sqrt{\lambda}} \begin{cases} e^{i\sqrt{\lambda}(y-x)}, & x < y \\ e^{i\sqrt{\lambda}(x-y)}, & x > y, \end{cases}$$

$$\Rightarrow G(x, y, \lambda) = \frac{i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda}|x-y|},$$

Consequences

(i) Because \bar{L} is analytic in $\lambda \notin [c, \infty)$,
 $G(x, y, \lambda)$ is also analytic there. In
addition, by construction, for each λ ,

$$(L - zI)^{-1} f = \int_{-\infty}^{\infty} G(x, y, \lambda) f(y) dy$$

defines a bdd. op. \Rightarrow If $\lambda \notin [c, \infty)$,
then $\lambda \in \rho(L)$. Also, $\sigma(L) \subseteq [c, \infty)$,

(ii) If $\lambda \in \lambda \in (-\infty, c)$, then E

(iii) If $\lambda < c$, then $E_n = 0$. This holds
because E_n is constant in any interval

in \mathbb{R} where m in \mathbb{R} where the interval
is in $\rho(L)$. Then $E_n = \text{const. in } (-\infty, c)$.

But, $\lim_{n \rightarrow -\infty} E_n = 0$, so $E_n = 0$. ~~Also, $E_n \rightarrow 0$~~

(5)

(iii) It is easy to show that L has no eigenvectors, ~~so~~ there are no eigenvalues; thus $\sigma_{\text{disc}}(L) = \{0\}$.

One can use this to show that $E_{A^+} = E_A$ in all A . Thus, if $t > 0$, ~~then~~

 ~~$E(A) + E(C)$~~

$$\frac{E_A + E_{A^-} - E_{A^+} + E_{A^+}}{2} = E_A - 0$$

$$\boxed{\frac{1}{2\pi i} \int_{\gamma}^{1-i} E_A = \frac{1}{2\pi i} \int_0^1 (G(x, y, t^+) - G(x, y, t^-)) dt,}$$

$$\Rightarrow E_A = \int_0^1 \frac{G(x, y, t^+) - G(x, y, t^-)}{2\pi i} dt.$$

3. Spectral measure.

Exercise ~~or~~ Let $t > 0$. Then,

$$\lim_{\varepsilon \downarrow 0} \overline{t \pm i\varepsilon} = \begin{cases} \overline{t} \\ -\overline{t} \end{cases},$$

$$\Rightarrow G(x, y, t^+) = \frac{i}{2\pi} e^{i\sqrt{t}|x-y|}, \quad G(x, y, t^-) = \frac{-i}{2\pi} e^{-i\sqrt{t}|x-y|}$$

$$\Rightarrow E_A = \frac{1}{2\pi i} \int_0^1 \frac{i}{2\sqrt{t}} \left(e^{i\sqrt{t}|x-y|} - e^{-i\sqrt{t}|x-y|} \right) dt.$$

$$\text{Let } u = \sqrt{t}, \quad E_A = \frac{1}{2\pi} \int_0^{\sqrt{A}} \cos(\sqrt{u}|x-y|) du.$$

Last time: Spectral measure.

Today: Fourier & other integral transforms.

1. Fourier transform. $\mathcal{L}u = -u''$, $D(\mathcal{L}) = \{u \in L^2 : u'' \in L^2\}$.

Earlier we showed that for $\lambda \geq 0$,

$$E_\lambda(u, y) = \frac{1}{\pi} \int_0^{1/\lambda} \underbrace{\cos((x-y)\zeta)}_{\text{even}} d\zeta,$$

$$= \frac{1}{2\pi} \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} \underbrace{\cos((x-y)\zeta)}_{\substack{\text{Cancels off abs. values}}} d\zeta$$

Projection based.

$$= \frac{1}{2\pi} \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} [\cos((x-y)\zeta) + i \underbrace{\sin((x-y)\zeta)}_{\substack{\text{odd, adds 0} \\ \text{to integrat}}}] d\zeta$$

$$\Rightarrow E_\lambda(x, y) = \frac{1}{2\pi} \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} e^{i(x-y)\zeta} d\zeta.$$

$$\Rightarrow E_\lambda f(x) = \frac{1}{2\pi} \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} e^{ix\zeta} \int_{-\infty}^{\infty} f(y) e^{-iy\zeta} dy$$

$f \in L^1 \cap L^2 \cap C(\mathbb{R})$

$$= : \hat{f}(\zeta)$$

$$\Rightarrow E_\lambda f(x) = \frac{1}{2\pi} \int_{-\frac{1}{\lambda}}^{\frac{1}{\lambda}} e^{ix\zeta} \hat{f}(\zeta) d\zeta$$

Now, $s\text{-}\lim_{\lambda \rightarrow +\infty} E_\lambda = I$, so $\lim_{\lambda \rightarrow +\infty} E_\lambda f = f \Rightarrow$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{ix\zeta} d\zeta$$

(2)

$$\text{Fourier transform} \quad \left\{ \begin{array}{l} \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx \\ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi \end{array} \right.$$

These integrals have restrictions on f . If $f \in L^2(\mathbb{R})$, they have to be interpreted as certain limits of integrals. As you might expect, there are also "almost everywhere" results, etc.

~~Proposition (see Folland, p. 218). Suppose that $f \in L^1(\mathbb{R})$ and that $f(x)$ is piecewise continuous. Then,~~

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi$$

Proposition. If $f \in L^1$ and is piecewise continuous, and at a jump x_0 , both left and right hand derivatives exist, * then

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi$$

Different Conventions

<u>Weiner</u> $F(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{iux} dx$ $f(x) = \int_{-\infty}^{\infty} F(u) e^{-iux} du$	<u>Symmetric</u> $\hat{f}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx$ $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi$
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* f is piecewise smooth.