Test III

Instructions: Show all work in your bluebook. Calculators that do linear algebra or calculus are not allowed.

1. (10 pts.) Find the eigenvalues and eigenvectors of $C = \begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}$.

Explain why $C$ is not diagonalizable.

2. In the following problem, $A = \begin{pmatrix} -4 & 3 \\ 3 & -12 \end{pmatrix}$.

(a) (15 pts.) Find the eigenvalues and eigenvectors of $A$ and a matrix $X$ such that $X^{-1}AX$ is diagonal.

(b) (10 pts.) Let $x(t) = [x_1(t), x_2(t)]^T$, where $x_1$ and $x_2$ are the equilibrium displacements for the masses in the spring system below. Newton’s law and Hooke’s law yield the equation $m\ddot{x} = kAx$. Use the answer to part (a) to find the normal modes for the system.


circ k m 3k m 9k

3. Consider the vector field $F(x) = 2xyi + (x^2 + 2yz)j + y^2k$.

(a) (15 pts.) Find the derivative matrix $D\!F$, the divergence $\nabla \cdot F$, and the curl $\nabla \times F$.

(b) (5 pts.) Is $F$ conservative – i.e., is $F = \nabla f$ for a scalar valued function?

4. (10 pts.) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}$, and $h = g \circ f : \mathbb{R}^2 \to \mathbb{R}$, where

$f(x, y) = (xy, 3x - 2y, x^2y)^T$ and $g(u, v, w) = uw + v^2 + w^2$

Use the chain rule to find the direction in which $h$ is increasing most rapidly at $x = y = 1$.

5. Let $(x, y) = T(u, v) = (u + 2uv, uv)$, and let $D^*$ be the $u$-$v$ rectangle $1 \leq u \leq 2, 0 \leq v \leq 1$.

(a) (5 pts.) Make a rough sketch of the image region $D = T(D^*)$.

(b) (10 pts.) Use Jacobi’s Theorem to change variables and compute the double integral $\int_D \cos \left(\frac{\pi y}{x - 2y}\right) \, dx \, dy$.

6. (20 pts.) Let $G(x) = 2yi - 3xj$ and let $C$ be the circle $x^2 + y^2 = 1$. Use the divergence form of Green’s Theorem to show that $\oint_C G \cdot \text{nds} = 0$, then verify this by direct computation of the line integral.