Test II

Instructions: Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are not allowed.

1. (10 pts.) Determine whether or not \( S = \{ f \in C[2,5] \mid f(2) = f(5) \} \) is a subspace of \( C[2,5] \).

2. (5 pts.) For the subsets of \( P_3 \) below, which cannot be linearly independent? Which cannot span? Briefly explain how you are getting your answers.
   (a) \( \{1 - 2x, x^2, x + x^3\} \)
   (b) \( \{x^2, x^3, 1 - 3x + 10x^2 - 5x^3\} \)
   (c) \( \{1, x^2 - 2, x^2 - 6\} \)
   (d) \( \{x - 3x^2, x^3, x^2 - 3x + 2, x - 1, 5x^2 - 7\} \)
   (e) \( \{1, x, x^3, x - x^2, 3x - 1, 2x^3\} \).

3. (10 pts.) Let \( B = \{1, e^x, e^{2x}\} \). Show that \( B \) is LI and find the coordinates of \( f(x) = (1 - 3e^x)^2 \).

4. (15 pts.) Find bases for the column space, null space, and row space of \( C \), and state the rank and nullity of \( C \). What should these sum to? Do they?
   \[
   C = \begin{pmatrix}
   1 & -2 & -1 & -3 \\
   -1 & 2 & 2 & 4 \\
   2 & -4 & -3 & -7
   \end{pmatrix}
   \]

5. Let \( L : P_2 \to P_2 \) be defined by \( L[p] = (x^2 - 1)p'' + (x + 3)p' - 4p \).
   (a) (5 pts.) Show that \( L \) is linear.
   (b) (5 pts.) Find the matrix \( A \) of \( L \) relative to the basis \( B = \{1, x, x^2\} \).
   (c) (5 pts.) Find a basis for the null space of \( L \). What is the rank of \( L \)?
6. **(15 pts.)** Consider the matrix $A$ below. Find the eigenvalues and eigenvectors of $A$. Also, find a diagonal matrix $\Lambda$ and an invertible matrix $S$ for which $A = S\Lambda S^{-1}$.

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$ 

7. **(15 pts.)** Consider the inner product $\langle f, g \rangle = \int_0^\pi f(x)g(x)dx$ on continuous functions $C[0, \pi]$. Use the Gram-Schmidt process to turn \{1, $\cos(x)$, $\cos^2(x)$\} into an orthogonal set relative to this inner product. (Hint: $\int \cos^n(x)dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x)dx$.)

8. Consider the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ on continuous functions $C[-1, 1]$.

   (a) **(5 pts.)** Find the Gram matrix $A$ for \{1, $x$, $x^2$\}.

   (b) **(10 pts.)** Use $A$ and the given inner product to find the best quadratic fit to $f(x) = |x|$.