Test II

Instructions: Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are not allowed.

1. (10 pts.) Let $P_3$ be the set of polynomials of degree 3 or less. Explain why $S = \{1 + x, 1 + x^2, 1 - x^2, 1 - x, x + x^3\}$ is not a basis for $P_3$. Then, extract a basis for $\text{Span}(S)$ from $S$. What is the dimension of this space? Does it coincide with $P_3$? Why? Be brief.

2. (15 pts.) Find bases for the column space, null space, and row space of $A$ below, and state the rank and nullity of $A$. What should these sum to? Do they?

$$A = \begin{pmatrix}
1 & -2 & -1 & -3 \\
-1 & 2 & 2 & 4 \\
2 & -4 & -3 & -7
\end{pmatrix}$$

3. Let $L : P_2 \rightarrow P_2$ be defined by $L[p] = (x^2 - 1)p'' + (x + 3)p' - 4p$.

(a) (5 pts.) Show that $L$ is linear.

(b) (5 pts.) Find the matrix $A$ of $L$ relative to the basis $B = \{1, x, x^2\}$.

(c) (5 pts.) Find bases for the null space of $L$ and the image of $L$. What are the nullity and rank of $L$?

4. Consider the bases for $P_2$ defined by

$$B = \{1, x, x^2\} \quad \text{and} \quad C = \{1, 1 + x, 1 - x - x^2\}.$$ 

(a) (5 pts.) Find the change of basis matrix $S$ that takes coördinates relative to $C$ into ones relative to $B$.

(b) (5 pts.) Use your answer to part 4a to find the change of basis matrix that takes $B$-coördinates into $C$-coördinates.

(c) (5 pts.) Use the matrix $A$ for $L$ from your answer to 3b and the change of bases matrices from your answers to 4a and 4b to find the matrix $A'$ for $L$ in the $C$-basis.
5. Consider the matrix \( A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \).

(a) (10 pts.) Find the eigenvalues and eigenvectors of \( A \).

(b) (5 pts.) Either find a diagonal matrix \( \Lambda \) and an invertible matrix \( S \) for which \( A = S\Lambda S^{-1} \) or explain why this is impossible.

6. Consider \( \langle f, g \rangle = \int_0^1 f(x)g(x)dx \), where \( f, g \) are in \( C[0, 1] \).

(a) (5 pts.) Show that \( \langle f, g \rangle \) is an inner product on \( C[0, 1] \).

(b) (10 pts.) You are given that \( \{1, \sqrt{3}(2x - 1)\} \) is an orthonormal set relative to this inner product. Using this, find the (continuous) least-squares linear fit to \( f(x) = \sqrt{x} \) on \([0, 1]\).

7. Let \( f(x) = |x|, -\pi \leq x \leq \pi \), and \( \langle g, h \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x)h(x)dx \), where \( g, h \) are in \( C[-\pi, \pi] \).

(a) (10 pts.) Find the Fourier series coefficients for \( f \).

(b) (5 pts.) Write down \( T_3 \). What is \( \|f - T_3\|^2 \)?