Test II

Instructions: Show all work in your bluebook. You may use a calculator for numerical computations. You may not use a graphing calculator or a calculator that can do symbolics.

1. **(20 pts.)** Let \( h(t) = \begin{cases} e^{-2t} & t \geq 0, \\ 0 & t < 0 \end{cases} \) and \( f(t) = \begin{cases} 1 & 0 \leq t \leq \pi, \\ 0 & t < 0 \text{ or } t > \pi. \end{cases} \) If \( H \) is the filter \( H[f] = h \ast f \), find \( H[f] \). Is \( H \) causal? Find the system function, \( \sqrt{2\pi h} \).

2. Let \( F_n \) be the DFT for \( n \)-periodic sequences (signals).
   
   (a) **(10 pts.)** Briefly describe the FFT. Why is it fast?
   
   (b) **(10 pts.)** Find \( F_4[a] \) if \( a = (-2, 1, 0, 1) \). Note: for \( n = 4 \), \( \omega = -i \).
   
   (c) **(10 pts.)** State the (circular) convolution theorem for the DFT, and use it and part 2b above to find the eigenvalues of the circulant matrix \( A \) whose first column is \( a^T = [-2 \ 1 \ 0 \ 1]^T \).

3. **(20 pts.)** Define multiresolution analysis (MRA). In the case of the Haar MRA, state or define the following items: the approximation spaces, \( V_j \), the scaling function \( \phi \), the two-scale (scaling) relation, and the wavelet \( \psi \). State what \( H \), \( L \) are in Fig. 1.

![Figure 1: Haar decomposition diagram.](image)

4. **(15 pts.)** Reconstruct \( f \in V_3 \), given these coefficients in its Haar wavelet decomposition:
   
   \[ a^1 = [3/2, -1] \quad b^1 = [-1, -3/2] \quad b^2 = [-3/2, -3/2, -1/2, -1/2]. \]
   
   The first entry in each list corresponds to \( k = 0 \). Sketch \( f \).

5. **(15 pts.)** Prove the Sampling Theorem: *Let \( f \) be band-limited. That is, suppose that \( \hat{f}(\lambda) \) is piecewise smooth, continuous, and that \( \hat{f}(\lambda) = 0 \) for \( |\lambda| > \Omega \), where \( \Omega \) is some fixed, positive frequency. Then \( f \) has the following series expansion:*
   
   \[
   f(t) = \sum_{j=-\infty}^{\infty} f(j\pi/\Omega) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}. \tag{1}
   \]