Test I

Instructions: *Show all work in your bluebook. You may use a calculator for numerical computations. You may* not *use a graphing calculator or a calculator that can do symbolics.*

1. Consider the function $f(x) := 1 - x$ defined on $0 \le x \le 1$. (Note: here $a = 1$, not π.)
   (a) **(15 pts.)** Find the Fourier *sine* series for f, and sketch three periods (period = 2) of the function to which it converges pointwise.
   (b) **(5 pts.)** Sketch three periods (period = 2) of the function to which the Fourier *cosine* series converges pointwise. (Do *not* compute the coefficients in the series.)
   (c) **(10 pts.)** Is either series uniformly convergent? If so, which? Why? Will either series exhibit the Gibbs' phenomenon? Briefly explain.

2. Let $g(x) = e^{2x}$ on $-\pi < x < \pi$
   (a) **(10 pts.)** Show that the complex form of the Fourier series for g is
   $$\frac{e^{2\pi} - e^{-2\pi}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2 - in} e^{inx}.$$
   (b) **(10 pts.)** Sum the series $\sum_{n=-\infty}^{\infty} \frac{1}{4+n^2}$ using the series above and Parseval's Theorem.

3. **(10 pts.)** Establish the Fourier transform property $\mathcal{F}[f'(x)](\lambda) = i\lambda\hat{f}(\lambda)$.

4. Let $a > 0$. Consider the rectangular pulse $h(t) = \begin{cases} 1 & -a < t < a \\ 1/2 & t = \pm a \\ 0 & |t| > a \end{cases}$.
   (a) **(10 pts.)** Find $\widehat{h}(\lambda)$.
   (b) **(10 pts.)** Find the convolution $h * h$.
   (c) **(5 pts.)** Find $\mathcal{F}[h * h](\lambda)$.

5. **(15 pts.)** Do *one* of the following.
   (a) State and prove the Riemann-Lebesgue Lemma in the case where $f(x)$ is continuously differentiable on the finite interval $[a, b]$.
   (b) Sketch the major steps in proof for the pointwise convergence of the Fourier series for a function f that is 2π-periodic, piecewise continuous, and has a piecewise continuous derivative. You may assume that f is continuous at the point x.

6. **(Bonus: 10 pts.)** Let g be as in problem **??**. Estimate the mean error $\|g - S_N(x)\|$, where $S_N(x) = \frac{e^{2\pi}-e^{-2\pi}}{2\pi} \sum_{n=-N}^{N} \frac{(-1)^n}{2-in} e^{inx}$. You may use any information you already obtained in problem **??**.

Fourier Transform Properties

1. $\hat{f}(\lambda) = \mathcal{F}[f](\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ix\lambda} dx.$

2. $f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\lambda) e^{ix\lambda} d\lambda.$

3. $\mathcal{F}[x^n f(x)](\lambda) = i^n \hat{f}^{(n)}(\lambda).$

4. $\mathcal{F}[f^{(n)}(x)](\lambda) = (i\lambda)^n \hat{f}(\lambda).$

5. $\mathcal{F}[f(x-a)](\lambda) = e^{-i\lambda a} \hat{f}(\lambda).$

6. $\mathcal{F}[f(bx)](\lambda) = \dfrac{1}{b} \hat{f}(\dfrac{\lambda}{b}).$

7. $\mathcal{F}[f * g] = \sqrt{2\pi} \hat{f}(\lambda) \hat{g}(\lambda)$

Integrals

1. $\int u dv = uv - \int v du$

2. $\int \dfrac{dt}{t} = \ln \left| t \right| + C$

3. $\int e^{at} dt = \frac{1}{a} e^{at} + C$

4. $\int t^n e^{at} dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1} e^{at} dt$

5. $\int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2+b^2} \left(a \cos(bt) + b \sin(bt) \right) + C$

6. $\int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2+b^2} \left(a \sin(bt) - b \cos(bt) \right) + C$

7. $\int t \sin(t) dt = \sin(t) - t \cos(t) + C$

8. $\int t \cos(t) dt = \cos(t) + t \sin(t) + C$