Test 1

Instructions: Show all work in your bluebook, except for problem 1. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are not allowed.

1. Place your answer in the space provided.

   (a) (5 pts.) Define the term inner product.

   (b) (5 pts.) Let $V$ be an inner product space and let $V_0$ a subspace of $V$. Define $V_0^\perp$.

   (c) (5 pts.) Let $\{f_n(x)\}$, where $x \in [0,1]$, be a sequence of functions. Define: $f_n$ converges uniformly to $f$ on $[0,1]$.

   (d) (5 pts.) Let $f(x) = \pi - x$, $0 \leq x \leq \pi$. Sketch three periods of the function to which its Fourier sine series converges pointwise.
2. (5 pts.) You are given that the $2\pi$-periodic function $g$, where $g(x) = \pi^2 - 3x^2$ on $-\pi \leq x \leq \pi$, has the Fourier series representation

$$g(x) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^2} \cos(nx). \quad (1)$$

Use (1) to find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. (15 pts.) For $n > 0$, let $f_n(t) = \begin{cases} 1, & 1 - 1/n \leq t \leq 1, \\ 0, & \text{otherwise}. \end{cases}$ Show that $f_n \to 0$ in $L^2[0,1]$. Show that $f_n$ does not converge to zero uniformly on $[0,1]$.

4. (20 pts.) Find the complex form of the Fourier series for $f(x) = xe^x$, $0 \leq x \leq 2\pi$.

5. (20 pts.) Let $f(x) = x$, $0 \leq x \leq \pi$. Find the Fourier cosine series for $f$.

6. (20 pts.) Do one of the following. (No extra credit for doing more.)

   (a) Prove this: Suppose $f$ is a continuous and $2\pi$-periodic function. Let $S_N$ of $f$ be the partial sum of the Fourier series for $f$. Show that

   $$S_N(x) = \int_{-\pi}^{\pi} f(x+u)P_N(u)du, \text{ where } P_N(u) = \frac{1}{\pi} \left( \frac{1}{2} + \sum_{n=1}^{N} \cos(nu) \right).$$

   (b) Prove this version of the Riemann-Lebesgue Lemma: If $f$ is continuously differentiable on the closed interval $[a,b]$, then

   $$\lim_{n \to \infty} \int_{a}^{b} f(x) \sin(nx) dx = 0.$$ 

   (c) Let $f(x) = e^{-x}$ and $V_0 = \text{span}\{1,2x-1\}$. Find the best least squares fit to $f$ in $L^2[0,1]$. 

2
Integrals

1. \( \int udv = uv - \int vdu \)

2. \( \int \frac{dt}{t} = \ln |t| + C \)

3. \( \int e^{at} dt = \frac{1}{a} e^{at} + C \)

4. \( \int t^n e^{at} dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1} e^{at} dt \)

5. \( \int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2 + b^2} (a \cos(bt) + b \sin(bt)) + C \)

6. \( \int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2 + b^2} (a \sin(bt) - b \cos(bt)) + C \)

7. \( \int t \sin(t) dt = \sin(t) - t \cos(t) + C \)

8. \( \int t \cos(t) dt = \cos(t) + t \sin(t) + C \)

9. \( \int \sin(at) dt = -\frac{1}{a} \cos(at) + C \)

10. \( \int \cos(at) dt = \frac{1}{a} \sin(at) + C \)

11. \( \int \tan(at) dt = \frac{1}{a} \ln |\sec(at)| + C \)

12. \( \int \cot(at) dt = \frac{1}{a} \ln |\sin(at)| + C \)

13. \( \int \sec(at) dt = \frac{1}{a} \ln |\sec(at) + \tan(at)| + C \)

14. \( \int \csc(at) dt = \frac{1}{a} \ln |\csc(at) - \cot(at)| + C \)

15. \( \int \frac{dt}{t^2 + a^2} = \frac{1}{a} \arctan(t/a) + C \)

16. \( \int \frac{dt}{t^2 - a^2} = \frac{1}{2a} \ln \left| \frac{t - a}{t + a} \right| + C \)