Test 1

Instructions: *Show all work in your bluebook, except for problem 1. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are* not *allowed.*

1. Place your answer in the space provided.

   (a) **(5 pts.)** Define the term *inner product*.

   (b) **(5 pts.)** Let V be an inner product space and let V_0 a subspace of V. Define $V_0^{\perp}$.

   (c) **(5 pts.)** Let $\{f_n(x)\}$, where $x \in [0,1]$, be a sequence of functions. Define: f_n converges *uniformly* to f on $[0,1]$.

   (d) **(5 pts.)** Let $f(x) = \pi - x$, $0 \leq x \leq \pi$. Sketch three periods of the function to which its Fourier sine series converges pointwise.

2. **(5 pts.)** You are *given* that the 2π-periodic function g, where $g(x) = \pi^2 - 3x^2$ on $-\pi \le x \le \pi$, has the Fourier series representation

$$g(x) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^2} \cos(nx). \tag{1}$$

Use (1) to find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. **(15 pts.)** For $n > 0$, let $f_n(t) = \begin{cases} 1, & 1 - 1/n \le t \le 1\,, \\ 0, & \text{otherwise.} \end{cases}$ Show that $f_n \to 0$ in $L^2[0,1]$. Show that f_n does *not* converge to zero uniformly on $[0,1]$.

4. **(20 pts.)** Find the *complex* form of the Fourier series for $f(x) = xe^x$, $0 \le x \le 2\pi$.

5. **(20 pts.)** Let $f(x) = x$, $0 \le x \le \pi$. Find the Fourier *cosine* series for f.

6. **(20 pts.)** Do *one* of the following. (No extra credit for doing more.)

   (a) Prove this: *Suppose f is a continuous and 2π-periodic function. Let S_N of f be the partial sum of the Fourier series for f. Show that*

   $$S_N(x) = \int_{-\pi}^{\pi} f(x+u) P_N(u) du, \text{where } P_N(u) = \frac{1}{\pi}\Big(\frac{1}{2} + \sum_{n=1}^{N} \cos(nu)\Big).$$

   (b) Prove this version of the Riemann-Lebesgue Lemma: *If f is continuously differentiable on the closed interval $[a,b]$, then*

   $$\lim_{n\to\infty} \int_a^b f(x) \sin(nx) dx = 0.$$

   (c) Let $f(x) = e^{-x}$ and $V_0 = \text{span}\{1, 2x-1\}$. Find the best least squares fit to f in $L^2[0,1]$.

Integrals

1. $\int u dv = uv - \int v du$
2. $\int \frac{dt}{t} = \ln\left|t\right| + C$
3. $\int e^{at} dt = \frac{1}{a} e^{at} + C$
4. $\int t^n e^{at} dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1} e^{at} dt$
5. $\int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2+b^2} \left(a \cos(bt) + b \sin(bt)\right) + C$
6. $\int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2+b^2} \left(a \sin(bt) - b \cos(bt)\right) + C$
7. $\int t \sin(t) dt = \sin(t) - t \cos(t) + C$
8. $\int t \cos(t) dt = \cos(t) + t \sin(t) + C$
9. $\int \sin(at) dt = -\frac{1}{a} \cos(at) + C$
10. $\int \cos(at) dt = \frac{1}{a} \sin(at) + C$
11. $\int \tan(at) dt = \frac{1}{a} \ln\left|\sec(at)\right| + C$
12. $\int \cot(at) dt = \frac{1}{a} \ln\left|\sin(at)\right| + C$
13. $\int \sec(at) dt = \frac{1}{a} \ln\left|\sec(at) + \tan(at)\right| + C$
14. $\int \csc(at) dt = \frac{1}{a} \ln\left|\csc(at) - \cot(at)\right| + C$
15. $\int \frac{dt}{t^2 + a^2} = \frac{1}{a} \arctan(t/a) + C$
16. $\int \frac{dt}{t^2 - a^2} = \frac{1}{2a} \ln\left|\frac{t-a}{t+a}\right| + C$