Test 1

Instructions: *Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are* not *allowed.*

1. Consider the function $f(x) := x$ defined on $0 \le x \le \pi$.
   (a) **(10 pts.)** Find the Fourier *cosine* series for f,
   (b) **(10 pts.)** Sketch three periods of the function to which the Fourier *cosine* series converges pointwise, and three periods of the function to which the Fourier *sine* series converges pointwise. (Do *not* compute the coefficients in the sine series.)
   (c) **(10 pts.)** Define the term *uniform convergence*. Is either series uniformly convergent? (Hint: use part 1b.) If so, which? Why? Will either series exhibit the Gibbs' phenomenon? Briefly explain.

2. Let $g(x) = e^{\frac{(1+i)}{2}x}$ on $-\pi < x < \pi$
   (a) **(15 pts.)** Show that the complex form of the Fourier series for g is
   $$g(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n i(e^{\pi/2} + e^{-\pi/2})}{\pi(1 + i(1-2n))} e^{inx}, \ -\pi < x < \pi.$$
   (b) **(10 pts.)** Sum the series $\sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2+1}$ using the series above and the complex form of Parseval's Theorem.

3. Let $f(x)$ be the "tent" function, $f(x) := \begin{cases} \pi - x, & \text{if } 0 \le x \le \pi \\ \pi + x, & \text{if } -\pi \le x \le 0 \\ 0, & \text{otherwise.} \end{cases}$
   (a) **(10 pts.)** Show that $\widehat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{1-\cos(\pi\lambda)}{\lambda^2}$.
   (b) **(10 pts.)** Prove Fourier transform property #4 for the $n = 1$ case.
   (c) **(10 pts.)** Find $\mathcal{F}\big[f'(x-\pi)\big]$.

4. **(15 pts.)** Do *one* of the following.
   (a) Let $V_N = \text{span}\{1, \cos(x), \sin(x), \ldots, \cos(Nx), \sin(Nx)\}$. Prove that the $L^2[-\pi, \pi]$ projection of $f \in L^2$ is the partial sum, $S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$.
   (b) Sketch the major steps in proof for the pointwise convergence, at a point x, of the Fourier series for a 2π-periodic, continuous function f for which $f'(x)$ exists.

5. **(Bonus: 10 pts.)** Let g be as in problem 2. Estimate the mean error $\|g - S_N(x)\|$, where $S_N(x) = \sum_{n=-N}^{N} \frac{(-1)^n i(e^{\pi/2}+e^{-\pi/2})}{\pi(1+i(1-2n))} e^{inx}$. You may use any information you already obtained in problem 2.

Fourier Transform Properties

1. $\hat{f}(\lambda) = \mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\lambda} dx.$
2. $\frac{f(x^+) + f(x^-)}{2} = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{ix\lambda} d\lambda.$
3. $\mathcal{F}[x^n f(x)](\lambda) = i^n \hat{f}^{(n)}(\lambda).$
4. $\mathcal{F}[f^{(n)}(x)](\lambda) = (i\lambda)^n \hat{f}(\lambda).$
5. $\mathcal{F}[f(x-a)](\lambda) = e^{-i\lambda a} \hat{f}(\lambda).$
6. $\mathcal{F}[f(bx)](\lambda) = \frac{1}{b} \hat{f}(\frac{\lambda}{b}).$
7. $\mathcal{F}[f * g] = \sqrt{2\pi} \hat{f}(\lambda) \hat{g}(\lambda)$

Integrals

1. $\int e^{at} dt = \frac{1}{a} e^{at} + C$
2. $\int t^n e^{at} dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1} e^{at} dt$
3. $\int t \sin(t) dt = \sin(t) - t\cos(t) + C$
4. $\int t \cos(t) dt = \cos(t) + t\sin(t) + C$
5. $\int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2 + b^2} (a\cos(bt) + b\sin(bt)) + C$
6. $\int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2 + b^2} (a\sin(bt) - b\cos(bt)) + C$
7. $\int \cos(at) \cos(bt) dt = \frac{\sin((a+b)t)}{2(a+b)} + \frac{\sin((a-b)t)}{2(a-b)} + C, \ a \neq b$
8. $\int \sin(at) \sin(bt) dt = \frac{\sin((a+b)t)}{2(a+b)} - \frac{\sin((a-b)t)}{2(a-b)} + C, \ a \neq b$
9. $\int \sin(at) \cos(bt) dt = -\frac{\cos((a+b)t)}{2(a+b)} - \frac{\cos((a-b)t)}{2(a-b)} + C, \ a \neq b$