Test 1

Instructions: Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are not allowed.

1. Consider the function \( f(x) := x \) defined on \( 0 \leq x \leq \pi \).
   (a) (10 pts.) Find the Fourier cosine series for \( f \).
   (b) (10 pts.) Sketch three periods of the function to which the Fourier cosine series converges pointwise, and three periods of the function to which the Fourier sine series converges pointwise. (Do not compute the coefficients in the sine series.)
   (c) (10 pts.) Define the term uniform convergence. Is either series uniformly convergent? (Hint: use part 1b.) If so, which? Why? Will either series exhibit the Gibbs’ phenomenon? Briefly explain.

2. Let \( g(x) = e^{i(\frac{1+i}{2})x} \) on \(-\pi < x < \pi\)
   (a) (15 pts.) Show that the complex form of the Fourier series for \( g \) is
   \[
   g(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n i (e^{n/2} + e^{-n/2})}{\pi(1+i(1-2n))} e^{inx}, \quad -\pi < x < \pi.
   \]
   (b) (10 pts.) Sum the series \( \sum_{n=-\infty}^{\infty} \frac{1}{12n-1+\pi} \) using the series above and the complex form of Parseval’s Theorem.

3. Let \( f(x) \) be the “tent” function, \( f(x) := \begin{cases} \pi - x, & \text{if } 0 \leq x \leq \pi \\ \pi + x, & \text{if } -\pi \leq x \leq 0 \\ 0, & \text{otherwise.} \end{cases} \)
   (a) (10 pts.) Show that \( \hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{1-\cos(\pi \lambda)}{\lambda^2} \).
   (b) (10 pts.) Prove Fourier transform property #4 for the \( n = 1 \) case.
   (c) (10 pts.) Find \( \mathcal{F}[f'(x - \pi)] \).

4. (15 pts.) Do one of the following.
   (a) Let \( V_N = \text{span}\{1, \cos(x), \sin(x), \ldots, \cos(Nx), \sin(Nx)\} \). Prove that the \( L^2[-\pi, \pi] \) projection of \( f \in L^2 \) is the partial sum, \( S_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx) \).
   (b) Sketch the major steps in proof for the pointwise convergence, at a point \( x \), of the Fourier series for a \( 2\pi \)-periodic, continuous function \( f \) for which \( f'(x) \) exists.

5. (Bonus: 10 pts.) Let \( g \) be as in problem 2. Estimate the mean error \( \|g - S_N(x)\| \), where \( S_N(x) = \sum_{n=-N}^{N} \frac{(-1)^n i (e^{n/2} + e^{-n/2})}{\pi(1+i(1-2n))} e^{inx} \). You may use any information you already obtained in problem 2.
Fourier Transform Properties

1. \( \hat{f}(\lambda) = F[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx. \)

2. \( \frac{f(x^+) + f(x^-)}{2} = F^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda)e^{ix\lambda}d\lambda. \)

3. \( F[x^n f(x)](\lambda) = i^n \hat{f}^{(n)}(\lambda). \)

4. \( F[f^{(n)}](\lambda) = (i\lambda)^n \hat{f}(\lambda). \)

5. \( F[f(x-a)](\lambda) = e^{-i\lambda a} \hat{f}(\lambda). \)

6. \( F[f(bx)](\lambda) = \frac{1}{b} \hat{f}\left(\frac{\lambda}{b}\right). \)

7. \( F[f * g] = \sqrt{2\pi} \hat{f}(\lambda)\hat{g}(\lambda) \)

Integrals

1. \( \int e^{at}dt = \frac{1}{a}e^{at} + C \)

2. \( \int t^n e^{at}dt = \frac{1}{a}t^n e^{at} - \frac{n}{a} \int t^{n-1}e^{at}dt \)

3. \( \int t \sin(t)dt = \sin(t) - t \cos(t) + C \)

4. \( \int t \cos(t)dt = \cos(t) + t \sin(t) + C \)

5. \( \int e^{at} \cos(bt)dt = \frac{e^{at}}{a^2 + b^2} (a \cos(bt) + b \sin(bt)) + C \)

6. \( \int e^{at} \sin(bt)dt = \frac{e^{at}}{a^2 + b^2} (a \sin(bt) - b \cos(bt)) + C \)

7. \( \int \cos(at) \cos(bt)dt = \frac{\sin((a+b)t)}{2(a+b)} + \frac{\sin((a-b)t)}{2(a-b)} + C, \ a \neq b \)

8. \( \int \sin(at) \sin(bt)dt = \frac{\sin((a+b)t)}{2(a+b)} - \frac{\sin((a-b)t)}{2(a-b)} + C, \ a \neq b \)

9. \( \int \sin(at) \cos(bt)dt = -\frac{\cos((a+b)t)}{2(a+b)} - \frac{\cos((a-b)t)}{2(a-b)} + C, \ a \neq b \)