Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.

Name__________________________________________________________
Instructions: Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.

Problem 1. Let $\mathcal{D}$ be the set of compactly supported $C^\infty$ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Give an example of a function in $\mathcal{D}$.
(c) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = \phi''(x)$ for some $\phi \in \mathcal{D}$ if and only if $\int_{-\infty}^{\infty} x\psi(x)dx = 0$.
(d) Use 2(c) to solve, in the distributional sense, the differential equation $u'' = 0$.

Problem 2. Consider the operator $Lu = -u''$ defined on functions in $L^2[0, \infty)$ having $u''$ in $L^2[0, \infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, $L$ has the domain $\mathcal{D}_L = \{u \in L^2[0, \infty) \mid u'' \in L^2[0, \infty) \text{ and } u'(0) = 0\}$.

(a) Find the Green’s function $G(x, \xi; z)$ for $-G'' - zG = \delta(x - \xi)$, with $G_x(0, \xi; z) = 0$.
(b) Employ the spectral theorem (Stone’s formula) to obtain the cosine transform formulas:

$$F(\mu) = \frac{2}{\pi} \int_0^{\infty} f(x) \cos(\mu x)dx \text{ and } f(x) = \int_0^{\infty} F(\mu) \cos(\mu x)d\mu.$$ 

Problem 3. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) Consider $K \in \mathcal{C}(\mathcal{H})$. Show that if $\{\phi_n\}_{n=0}^\infty$ is an orthonormal set in $\mathcal{H}$, then $\lim_{n \to \infty} K\phi_n = 0$.
(b) Suppose that $K \in \mathcal{C}(\mathcal{H})$ is self adjoint.

(i) Show that $\sigma(K)$ (the spectrum) consists only of eigenvalues, together with 0, and that the only limit point of $\sigma(K)$ is 0.
(ii) Given that $\|K\| = \sup_{\|u\|=1} |\langle Ku, u \rangle|$, show that either $\|K\|$ or $-\|K\|$ (or possibly both) is an eigenvalue of $K$, and that the corresponding eigenspace is finite dimensional.

Problem 4. Suppose that $f(x)$ is $2\pi$-periodic function in $C^{(m)}(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and $2\pi$-periodic. Here $m > 0$ is a fixed integer. Let $c_k$ denote the $k^{th}$ (complex) Fourier coefficient for $f$, and let $c_k^{(j)}$ denote the $k^{th}$ (complex) Fourier coefficient for $f^{(j)}$.

(a) Prove that $c_k^{(j)} = (ik)^j c_k$, $j = 0, \ldots, m+1$. (Note: using term by term differentiation of the Fourier series assumes what you want to prove.)
(b) For $k \neq 0$, show that $c_k$ satisfies the bound

$$|c_k| \leq \frac{1}{2\pi |k|^{m+1}} \|f^{(m+1)}\|_{L^1[0, 2\pi]},$$

(c) Let $f_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ be the $n^{th}$ partial sum of the Fourier series for $f$, $n \geq 1$. Show that

$$\|f - f_n\|_{L^2[0, 2\pi]} \leq C \frac{\|f^{(m+1)}\|_{L^1[0, 2\pi]}}{n^{m+\frac{1}{2}}},$$

where $C$ is independent of $f$ and $n$. 