APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

August 6, 2019

Applied Analysis Part, 2 hours

Name: ____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let $f \in C[0,1]$, $\delta > 0$, and $\omega(f,\delta)$ be the modulus of continuity for f.

(a) Let $\Delta = \{x_0 = 0 < x_1 < \cdots < x_n = 1\}$ be a knot sequence with norm $\|\Delta\| = \max |x_j - x_{j+1}|$, $j = 0,\ldots,n-1$. If s_f is the linear spline that interpolates f at the x_j's, show that $\|f - s_f\|_\infty \le \omega(f, \|\Delta\|)$.

(b) Using part (a) and the fact that the continuous functions are dense in $L^1[0,1]$, prove the Riemann-Lebesgue Lemma: $\lim_{|\lambda|\to\infty} \int_0^1 g(x)e^{i\lambda x}dx = 0$, for all $g \in L^1[0,1]$.

Problem 2. Let $\mathcal{D}$ be the set of compactly supported C^∞ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.

(b) Consider a function $f \in C^{(1)}(\mathbb{R})$ such that both f and f' are in $L^1(\mathbb{R})$, and $\int_{\mathbb{R}} f(x)dx = 1$. Define the sequence of functions $\{T_n(x) := n^2 f'(nx) \colon n = 1,2,\ldots\}$. Show that, in the sense of distributions — i.e., in $\mathcal{D}'$ —, T_n converges to δ'.

Problem 3. Let L be a closed, densely defined (possibly unbounded) linear operator on a Hilbert space $\mathcal{H}$, and let the range of L be dense in $\mathcal{H}$.

(a) Show that if there exists $C > 0$ such that $\|Lf\| \ge C\|f\|$ for all $f \in \mathcal{D}$, then L^{-1} is bounded.

(b) Use (a) to show that if $L = L^*$, then the spectrum of L is contained in $\mathbb{R}$.

Problem 4. Consider the boundary problem below::

$$L[u] = \frac{d}{dx}\left(x\frac{du}{dx}\right) = f, \text{ where } \mathcal{D} = \{u \in L^2[1,e] \colon Lu \in L^2[1,e],\ u'(1) = 0,\ u(e) = 0\},$$

(a) Find the Green's function $g(x,y)$ for the problem, given that $1, \log(x)$ solve $L[u] = 0$.

(b) Show that $Kf(x) = \int_1^e g(x,y)f(y)dy$ is self adjoint, and briefly explain why it's compact. Show *directly* from the spectral theory for compact operators that the orthonormal set of eigenfunctions for L is complete in $L^2[1,e]$. (Do not solve the eigenvalue problem.)