APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

August 7, 2020

Applied Analysis Part, 2 hours

Name: ____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let T be a bounded, invertible operator on a Hilbert space $\mathcal{H}$, K be a compact operator on $\mathcal{H}$, and $L = T - \lambda K$, $\lambda \in \mathbb{C}$. Show that the range of L is closed.

Problem 2. Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be a set of polynomials orthogonal with respect to a positive weight function $w \in C[0,1]$. Assume that the degree of ϕ_n is n, and that coefficient of x^n in $\phi_n(x)$ is $k_n > 0$.

(a) Show that ϕ_n is orthogonal to all polynomials of degree $n-1$ or less.
(b) Show that the set $\{\phi_n(x)\}_{n=0}^{\infty}$ is the same, up to multiples, as the one gotten by using the Gram-Schmidt process.
(c) Show that the polynomials satisfy the recurrence relation below; find A_n in terms of the k_n's.

$$\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) + C_n \phi_{n-1}(x)$$

Problem 3. Consider the operator $Lu = -u''$, where $\mathcal{D}_L := \{u \in L^2(\mathbb{R}) : Lu \in L^2(\mathbb{R})\}$.

(a) Show that L is self adjoint and positive definite.
(b) Find the Green's function

$$L_x g(x,y) - \lambda g(x,y) = \delta(x-y), \ \lambda \notin \ [0,\infty)$$

for L. Hint: the left and right boundary conditions are that $g(x,y)$ be in $L^2(-\infty, y)$ and in $L^2(y,\infty)$, respectively. Also, choose $\mathrm{Im}\sqrt{\lambda} > 0$.
(c) Is $g(x,y)$ a Hilbert-Schmidt kernel? Prove your answer.

Problem 4. Let A be a real $n \times n$ self-adjoint matrix, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) State and prove the Courant-Fischer Minimax Theorem for A.
(b) Use it to show $\lambda_2 < 0$ for

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$