Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 5, 2022

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and in $1\mathcal{D}'$.
(b) Give an example of a function in $\mathcal{D}$.
(c) Show that $\psi \in \mathcal{D}$ has the form $\psi(x) = (x\phi(x))'$ for some $\phi \in \mathcal{D}$ if and only if $\int_{-\infty}^{\infty} \psi(x)dx = \int_0^{\infty} \psi(x)dx = 0$.
(d) Use 2(c) to show that if $T \in \mathcal{D}'$ satisfies $xT'(x) = 0$, then $T(x) = \delta(x)$.

Problem 2. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the Banach space of bounded operators on $\mathcal{H}$. In addition, let $\mathcal{C}(\mathcal{H})$ be the subspace of $\mathcal{B}(\mathcal{H})$ comprising the compact operators on $\mathcal{H}$.

(a) Show that $\mathcal{C}(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{H})$.
(b) Show that if $L \in \mathcal{B}(\mathcal{H})$ has finite rank (i.e., the range of L is finite dimensional), then L is compact.
(c) Let $\mathcal{H} = L^2[0,1]$. Show if $k(x,y)$ is a Hilbert Schmidt kernel, then $Ku(x) = \int_0^1 k(x,y)u(y)dy$ is compact. (You may assume that $K \in \mathcal{B}(\mathcal{H})$.)

Problem 3. Let $\mathcal{P}$ be the set of all polynomials in x.

(a) State and sketch a proof of the Weierstrass approximation theorem.
(b) Show that $\mathcal{P}$ is dense in $L^2[0,1]$.
(c) Let $\mathcal{S} := \{p_n\}_{n=0}^{\infty}$ be the orthonormal set of polynomials obtained from $\mathcal{P}$ via the Gram-Schmidt process. Show that $\mathcal{S}$ is a complete set in $L^2[0,1]$.

Problem 4. Consider the self-adjoint operator $Lu = -u''$ having the domain $\mathcal{D}_L := \{u \in L^2[0,\infty) : u(0) = 0 \ \& \ Lu \in L^2[0,\infty)\}$.

(a) Show that L has no eigenvalues $\lambda \in \mathbb{C} \setminus [0,\infty)$.
(a) Assume that $\mathrm{Im}\sqrt{\lambda} > 0$. Find the Green's function for L,

$$L_x g(x,y) - \lambda g(x,y) = \delta(x-y), \ \lambda \notin \ [0,\infty).$$

(c) Is $g(x,y)$ a Hilbert-Schmidt kernel? Prove your answer.