Name: ________________________________

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.

Instructions: Do any four problems. Show all work clearly. State the problem that you are skipping. No extra credit for doing all five.

Problem 1. Let $A$ be an $n \times n$ self-adjoint matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) State the Courant-Fischer mini-max theorem.

(b) Let $B = [b_1 \ b_2]$ be a real $n \times 2$ matrix, with $b_1, b_2$ being linearly independent. Assume that $\|x\| = 1$. If $q(x) = x^T Ax$ and $\hat{q}(x) = q(x)|_{B^T x = 0}$, show that

$$\lambda_3 \leq \max_{\|x\|=1} \hat{q}(x) \leq \lambda_1.$$  

Problem 2. A sequence $\{f_n\}$ in $H$ is said to be weakly convergent to $f \in H$ if and only if

$$\lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle$$

for every $g \in H$. When this happens, we write $f = \text{w-lim}_{n \to \infty} f_n$. One can show that every weakly convergent sequence is bounded.

(a) Let $\{\phi_n\}_{n=1}^\infty$ be any orthonormal sequence. Show that $\text{w-lim}_{n \to \infty} \phi_n = 0$. (Hint: use Bessel’s inequality.)

(b) Let $K$ be a compact linear operator on a Hilbert space $H$. Show that if $\text{w-lim}_{n \to \infty} f_n = f$, then $\lim_{n \to \infty} K f_n = K f$.

(c) Define $\rho(K)$, the resolvent set for $K$, and $\sigma(K)$, the spectrum of $K$. Use (a) and (b) to show that $0 \in \sigma(K)$.

Problem 3. Let $J[y] := \int_0^1 \left( \frac{1}{2} y'^2 + yy' + y + y \right) dx$. Find the extremal of $J$ that satisfies natural boundary conditions at $x = 0$ and $x = 1$.

Problem 4. Consider the operator $Lu = x^2 u'' - xu'$ with domain $D_L := \{ u \in L^2[1, 2] : Lu \in L^2[1, 2], \ u(1) = 0 \ & \ u'(2) = 0 \}$. You are given that the homogenous solutions of $Lu = 0$ are 1 and $x^2$, neither of which is in $D_L$.

(a) Compute the adjoint $L^*$, along with the adjoint boundary conditions. Is $L$ self adjoint?

(b) Compute the Green’s function for $L$.

(c) Is $L^{-1}$ compact? Justify your answer.

Problem 5. State and sketch a proof of the Weierstrass Approximation Theorem.