APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER
January 9, 2020
Applied Analysis Part, 2 hours

Name: ____________________________

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.

Instructions: Do any three problems. Show all work clearly. State the problem that you are skipping. No extra credit for doing all four.

Problem 1. Consider $F(x) := \frac{x}{2} + \frac{1}{x}$, $1 \leq x \leq 2$.
   
   (a) State and prove the Contraction Mapping Theorem.
   (b) Show that $F : [1, 2] \to [1, 2]$, that it is Lipschitz continuous on $[1, 2]$, with Lipschitz constant less than or equal to 1/2.
   (c) Obviously, the fixed point is $\sqrt{2}$. If $x_0 = 2$, estimate the number of iterations needed to come within 0.001 of $\sqrt{2}$.

Problem 2. Let $p \in C^{(2)}[0, 1], q \in C[0, 1]$ be positive on $[0, 1]$. Consider the operator $Lu = -(pu')' + qu$,
where $D_L := \{u \in L^2[0, 1] : Lu \in L^2[0, 1], u(0) = 0 \text{ & } u'(1) = 0\}$.

   (a) Show that $L$ is self adjoint and positive definite.
   (b) Explain why the Green’s function $g(x, y)$ exists for this problem.
   (b) Prove that the eigenfunctions of $L$ contain a complete, orthonormal set with respect to $L^2[0, 1]$.

Problem 3. Let $\mathcal{H}$ be a Hilbert space, $C(\mathcal{H})$ the compact operators $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})$ be the bounded operators on $\mathcal{H}$.

   (a) Prove that $C(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$.
   (b) Let $\mathcal{H} = L^2[0, 1]$. Use the result above to show that a Hilbert-Schmidt operator $Ku(x) = \int_0^1 k(x, y)u(y)dy, k \in L^2([0, 1] \times [0, 1])$ is compact.

Problem 4. Let $\mathcal{S}$ be Schwartz space and $\mathcal{S}'$ be the space of tempered distributions. The Fourier transform convention used here is $\mathcal{F}[f](\omega) = \hat{f}(\omega) := \int_{\mathbb{R}} f(t)e^{i\omega t}dt, \mathcal{F}^{-1}[\hat{f}](x) = f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega)e^{-i\omega t}d\omega$.

   (a) Sketch a proof: The Fourier transform $\mathcal{F}$ is a continuous linear operator mapping $\mathcal{S}$ into itself.
   (b) Use the previous result to show that $\langle \mathcal{F}[T](x), \phi(x) \rangle := \langle T(x), \mathcal{F}[\phi](x) \rangle$ implies $\mathcal{F}[T] \in \mathcal{S}'$.
   (c) You are given that if $T \in \mathcal{S}',$ then $\hat{T}^{(k)} = (-i\omega)^k \hat{T}$, where $k = 1, 2, \ldots$. Let $T$ be the tent function $T(x) = 1 - |x|$, $|x| \leq 1$, and $T(x) = 0$ otherwise. Find $\hat{T}$. (Hint: What is $T''$?)

1Here we are defining $\langle f, g \rangle := \int_{\mathbb{R}} f(x)g(x)dx$. Note that there is no complex conjugate in this definition of $\langle f, g \rangle$. 