# APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

**January 9, 2020**

**Applied Analysis Part, 2 hours**

**Name:** ______________________________

**Policy on misprints:** The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

**Instructions:** *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

**Problem 1.** Consider $F(x) := \frac{x}{2} + \frac{1}{x}$, $1 \le x \le 2$.

(a) State and prove the Contraction Mapping Theorem.

(b) Show that $F : [1, 2] \to [1, 2]$, that it is Lipschitz continuous on $[1, 2]$, with Lipschitz constant less than or equal to $1/2$.

(c) Obviously, the fixed point is $\sqrt{2}$. If $x_0 = 2$, estimate the number of iterations needed to come within 0.001 of $\sqrt{2}$.

**Problem 2.** Let $p \in C^{(2)}[0, 1]$, $q \in C[0, 1]$ be positive on $[0, 1]$. Consider the operator $Lu = -(pu')' + qu$, where $\mathcal{D}_L := \{u \in L^2[0, 1]] : Lu \in L^2[0, 1]],\ u(0) = 0\ \&\ u'(1) = 0\}$.

(a) Show that $L$ is self adjoint and positive definite.

(b) Explain why the Green's function $g(x, y)$ exists for this problem.

(b) Prove that the eigenfunctions of $L$ contain a complete, orthonormal set with respect to $L^2[0, 1]$.

**Problem 3.** Let $\mathcal{H}$ be a Hilbert space, $\mathcal{C}(\mathcal{H})$ the compact operators $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})$ be the bounded operators on $\mathcal{H}$.

(a) Prove that $\mathcal{C}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$.

(b) Let $\mathcal{H} = L^2[0, 1]$. Use the result above to show that a Hilbert-Schmidt operator $Ku(x) = \int_0^1 k(x, y)u(y)dy$, $k \in L^2([0, 1] \times [0, 1])$ is compact.

**Problem 4.** Let $\mathcal{S}$ be Schwartz space and $\mathcal{S}'$ be the space of tempered distributions. The Fourier transform convention used here is $\mathcal{F}[f](\omega) = \widehat{f}(\omega) := \int_{\mathbb{R}} f(t)e^{i\omega t}dt$, $\mathcal{F}^{-1}[\widehat{f}](x) = f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)e^{-i\omega t}d\omega$.

(a) Sketch a proof: *The Fourier transform $\mathcal{F}$ is a continuous linear operator mapping $\mathcal{S}$ into itself.*

(b) Use the previous result to show that[1] $\langle \mathcal{F}[T](x), \phi(x)\rangle := \langle T(x), \mathcal{F}[\phi](x)\rangle$ implies $\mathcal{F}[T] \in \mathcal{S}'$.

(c) You are **given** that if $T \in \mathcal{S}'$, then $\widehat{T^{(k)}} = (-i\omega)^k \widehat{T}$, where $k = 1, 2, \ldots$. Let $T$ be the tent function $T(x) = 1 - |x|$, $|x| \le 1$, and $T(x) = 0$ otherwise. Find $\widehat{T}$. (Hint: What is $T''$?)

[1] Here we are defining $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)dx$. Note that there is no complex conjugate in this definition of $\langle f, g\rangle$.