Applied Analysis Part

January 11, 2022

Name: ______________________________

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let ψ_j and ϕ_j, $j = 1, \dots, n$, be in $L^2[0,1]$. Assume the sets $\{\psi_j\}_{j=1}^n$ and $\{\phi_j\}_{j=1}^n$ are linearly independent. Consider the finite rank kernel $k(x,y) = \sum_{j=1}^n \psi_j(x)\overline{\phi}_j(y)$ and let $Ku(x) = \int_0^1 k(x,y)u(y)dy$. You are given that K is compact.

(a) State and prove the Fredholm Alternative.
(b) State the Closed Range Theorem.
(c) Show that the equation $(I - \lambda K)u = f$ has an L^2-solution for all $f \in L^2[0,1]$ if and only if $1/\bar{\lambda}$ is *not* an eigenvalue of the matrix A, where $A_{jk} = \langle \phi_j, \psi_k \rangle$.

Problem 2. Let both $K \in \mathcal{C}(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{H})$ be self adjoint.

(a) Show that $\|L\|_{op} = \sup_{\|u\|=1} |\langle Lu, u\rangle|$. (Hint: look at $\langle L(u+v), u+v\rangle - \langle L(u-v), u-v\rangle$, then apply the polarization identity.)
(b) Prove this: *Either* $\|K\|$ *or* $-\|K\|$ *is an eigenvalue of* K.
(c) Let $\mathcal{H} = L^2[01]$ and define the operator $M : L^2[0,1] \to L^2[0,1]$ by $Mu(x) = xu(x)$. Show that $\|M\|_{op} = 1$. Is M compact? Prove your answer.

Problem 3. Suppose that $Lu = u'' + \lambda u$, with $\mathrm{Dom}(L) = \{u \in L^2(-\infty,\infty) : Lu \in L^2(-\infty,\infty)\}$, where $\lambda \in \mathbb{C} \setminus [0,\infty)$. In addition, choose $\mathrm{Im}\sqrt{\lambda} > 0$. Show that the Green's function for L is given by

$$g(x,y,\lambda) = \tfrac{-i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda}|x-y|}$$

Problem 4. Consider the functions ϕ and ψ defined below:

$$\phi(x) = \begin{cases} (|x|-1)^2(2|x|+1) & |x| \le 1 \\ 0 & |x| > 1 \end{cases}, \qquad \psi(x) = \begin{cases} x(|x|-1)^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}.$$

Recall that for $n \ge 2$ and $0 \le j \le n$, the functions $\phi_j(x) := \phi(nx - j)$ and $\psi_j(x) := \frac{1}{n}\psi(nx - j)$ satisfy $\phi_j(k/n) = \delta_{j,k}$, $\phi_j'(k/n) = 0$, $\psi_j(k/n) = 0$ and $\psi_j'(k/n) = \delta_{j,k}$. In addition, the set $\{\phi_j, \psi_j\}_{j=0}^n$ is a basis for the finite element space $S^{\frac{1}{n}}(3,1)$.

(a) Let $S_0^{1/n}(3,1) = \{s \in S^{\frac{1}{n}}(3,1) : s(0) = s(1) = 0\}$. Show that $\langle u, v\rangle = \int_0^1 u''v''dx$ defines an inner product on $S_0^{1/n}(3,1)$, and that $\{\phi_j\}_{j=1}^{n-1} \cup \{\psi_j\}_{j=0}^n$ is a basis for $S_0^{1/n}(3,1)$.
(b) Show that $\langle \psi_j, \psi_k\rangle = 0$ for all j,k such that $|j-k| > 1$.
(c) Show that $\mathrm{argmin}\{\|s\| : s \in S_0^{\frac{1}{n}}, s(j/n) = f_j,\ j = 1, \dots n-1\}$ is given by $s(x) = \sum_{j=1}^{n-1} f_j\phi_j(x) - \sum_{j=0}^n \alpha_j\psi_j(x)$, where α_j's satisfy a tridiagonal system. Why is this system invertible?