APPLIED/NUMERICAL QUALIFIER
NUMERICAL ANALYSIS PART

August 7, 2020.

For Problem 1, assume that Ω is a convex polygonal domain in $\mathbb{R}^2$. Let $\{\mathcal{T}_h\}_{h\in(0,1)}$ be a set of shape-regular triangulations of Ω. For each h, let V_h be the set of continuous functions which are piecewise linear with respect to $\mathcal{T}_h$ and vanish on $\partial\Omega$. Define $D(\cdot,\cdot)$ to be the Dirichlet inner product,

$$D(v,w) := \int_\Omega \nabla v \cdot \nabla w \, dx, \qquad \text{for all } v,w \in H_0^1(\Omega).$$

Let P_h denote the elliptic projection, i.e., $P_h : H_0^1(\Omega) \to V_h$ is defined by $P_h v := w_h$ where $w_h \in V_h$ is the unique solution to the problem

$$D(w_h,\theta_h) = D(v,\theta_h) \qquad \text{for all } \theta_h \in V_h.$$

Problem 1. Let f be in $C^0([0,T];L^2(\Omega))$ and u_0 be in $H_0^1(\Omega)\cap H^2(\Omega)$. Consider the parabolic initial value problem: $u:[0,T]\to H_0^1(\Omega)$ defined by

$$\partial_t u(t) - \Delta u(t) = f(t), \quad t\in(0,T], \quad u(0)=u_0 \in H_0^1(\Omega)\cap H^2(\Omega). \tag{1.1}$$

Accept as a fact that the above problem has a unique solution with regularity $u \in C^1([0,T];H^2(\Omega)\cap H_0^1(\Omega))$. Let $k>0$ and consider the fully discrete approximation based on backward Euler time stepping, i.e., $U^j \approx u(t_j)$ where $t_j = jk$ satisfies $U^0 = u_{h,0}\in V_h$ and for $j=0,1,\ldots,$

$$\left(\frac{U^{j+1}-U^j}{k},\phi_h\right) + D(U^{j+1},\phi_h) = (f(t_{j+1}),\phi_h), \qquad \text{for all } \phi_h \in V_h. \tag{1.2}$$

Here $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

(a) State an estimate for the error $\|(I-P_h)v\|_{L^2(\Omega)}$ for all $h\in(0,1)$ and v in $H_0^1(\Omega)\cap H^2(\Omega)$.

(b) Show that the following holds true for all $\ell \ge 1$:

$$\|U^\ell\|_{L^2(\Omega)} \le \|U^0\|_{L^2(\Omega)} + k\sum_{j=1}^{\ell} \|f(t_j)\|_{L^2(\Omega)}. \tag{1.3}$$

(c) Let $\eta(t) := u(t) - P_h(u(t))$ for all $t\in[0,T]$. Show that there exists C such that for all $h\in(0,1)$,

$$\|\eta(t)\|_{L^2(\Omega)} \le Ch^2\left[\int_0^t \|u_\tau\|_{H^2(\Omega)}\,d\tau + \|u_0\|_{H^2(\Omega)}\right].$$

You may use the following inequality without proof:

$$\left\|\int_0^t \partial_\tau u\,d\tau\right\|_{H^2(\Omega)} \le \int_0^t \|\partial_\tau u\|_{H^2(\Omega)}\,d\tau. \tag{1.4}$$

(d) Let $\theta^j := P_h u(t_j) - U^j$ for all $j = 0, 1 \ldots$. Notice that $\{\theta^j\} \subset V_h$. Show that

$$\left(\frac{\theta^{j+1} - \theta^j}{k}, \phi_h\right) + D(\theta^{j+1}, \phi_h) = (w^{j+1}, \phi_h), \qquad \text{for all } \phi_h \in V_h \text{ and } j = 0, 1, \ldots \tag{1.5}$$

where

$$w^{j+1} := \frac{1}{k}\int_{t_j}^{t_{j+1}} \partial_\tau P_h(u(\tau))d\tau - \partial_t u(t_{j+1}). \tag{1.6}$$

Problem 2. Consider the system

$$\begin{aligned} -\Delta u - v &= f \\ u - \Delta v &= g \end{aligned} \tag{2.1}$$

in a bounded, smooth domain Ω in $\mathbb{R}^n$, with boundary conditions $u = v = 0$ on $\partial\Omega$. (You may use that $\phi \mapsto \|\nabla\phi\|_{(L^2(\Omega))^n}$ is a norm on the Hilbert space $H_0^1(\Omega)$.)

(a) Derive a weak formulation of the system (2.1), using suitable test functions (ϕ, ψ) for each equation resulting in a problem of the form: find (u, v) satisfying

$$a\big((u, v), (\phi, \psi)\big) = (f, \phi) + (g, \psi) \tag{2.2}$$

and explicitly define $a(\cdot, \cdot)$ and the function spaces for u, v, ϕ and ψ appearing in (2.2).

(b) Show that the weak formulation (2.2) has a unique solution.

(c) Let $d > 0$ and $\Omega_d := (-d, d)^2$. Show that there exists a positive number c such that the following holds true for every $d > 0$ and every $u \in H_0^1(\Omega_d)$. (You may use that $C_0^1(\Omega_d)$ is dense in $H_0^1(\Omega_d)$.)

$$\|u\|^2_{L^2(\Omega_d)} \leq cd^2\|\nabla u\|^2_{(L^2(\Omega_d))^2}. \tag{2.3}$$

(d) Now change the second minus sign in the first equation of (2.1) to a plus sign. Use (2.3) to show stability for the modified equation on Ω_d provided that d is sufficiently small.

Problem 3. Let $a = y_0 < y_1 < \ldots < y_N = b$ be a partition of the interval $[a, b]$ and $J_i = [y_{i-1}, y_i]$ denote the i'th subinterval. Set $y_{i+1/2} := (y_{i+1} + y_i)/2$ and $h := \max_{1 \leq i \leq N}\{y_i - y_{i-1}\}$. Define V_h to be the set of functions $f \in C^1(a, b)$ which are piecewise quadratic with respect to the composite mesh $y_0 < y_{1/2} < y_1 < y_{3/2} < \cdots < y_N$.

(a) Show that a function $f \in V_h$ restricted to J_i is uniquely determined by its values:

$$f(y_{i-1}), f'(y_{i-1}), f(y_i), f'(y_i).$$

Hint: You can assume without proof that this problem can be reduced to the reference element case, i.e., **show that a function $f \in C^1(0, 1)$ which is piecewise quadratic with respect to the intervals $[0, 1/2]$ and $[1/2, 1]$ is uniquely determined by its values:**

$$f(0), f'(0), f(1), f'(1).$$

(b) Construct the local shape functions associated with the degrees of freedom on the interval $[0, 1]$ defined in Part (a).

(c) What is the dimension of V_h (explain your answer)?