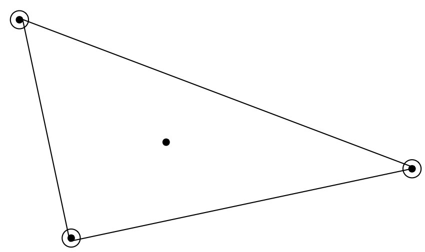
NUMERICAL ANALYSIS QUALIFIER

August, 2021

Problem 1. Consider the C^0 Hermite cubic finite element $(T, \mathbb{P}^3, \Sigma)$, where

$T \subset \mathbb{R}^2$ is a triangle with vertices v_1, v_2, v_3 and barycenter b,
$\mathbb{P}^3$ is the set of polynomials of degree 3 or less on T,
$\Sigma = \{p(v_i), p(b), \nabla p(v_i),\ i = 1, 2, 3\}$.

The diagram for the degrees of freedom is below (dots are function evaluations, circles are gradient evaluations). Show that $(T, \mathbb{P}^3, \Sigma)$ is a finite element.



Hints:

(1) You may assume without loss of generality that T is the unit triangle if you wish.
(2) Each gradient evaluation $\nabla p(v_i)$ yields *two* degrees of freedom which may be taken to be any two directional derivatives in independent directions.
(3) You may use the following elementary factorization result without proof:

If $p \in \mathbb{P}^k$ and $p = 0$ on the line $L(x, y) = 0$, then $p = Lp_{k-1}$ with $p_{k-1} \in \mathbb{P}^{k-1}$.

Problem 2. Consider the boundary value problem

$$\begin{aligned} &-u''(x) - \alpha u(x) = f(x), \quad 0 < x < 1, \\ &u(0) = 0,\ u'(1) = 0, \end{aligned} \tag{2.1}$$

where $f(x)$ is a given function on $(0, 1)$ and $\alpha > 0$ is a given constant. *Note:* Below you may assume:

(1) The validity of an appropriate Poincaré inequality.
(2) Approximation (interpolation) error bounds for the finite element spaces you define below.

However, be sure to correctly and clearly state these results with appropriate hypotheses before using them.

(a) Give a weak formulation of this problem. As part of deriving the weak formulation, be sure to define an appropriate variational space V.

(b) Prove that the corresponding bilinear form is coercive on V. This result is only valid for a restricted range of values of the parameter α. Clearly state for which α your result holds, and explicitly include dependence on α and the Poincaré constant in your coercivity constant.
(c) Set up a finite dimensional space $V_h \subset V$ of piece-wise polynomial functions of degree k over a uniform partition of $(0, 1)$. Introduce the Galerkin finite element method for the problem (2.1) for V_h. State (but do not prove) an error estimate in the V-norm assuming that $u \in H^{k+1}(0, 1)$.

(d) Assuming "full regularity" and using a duality argument **prove** the following estimate for the error of the Galerkin solution u_h:

$$\|u - u_h\|_{L^2} \leq Ch^{k+1}\|u^{(k+1)}\|_{L^2}. \tag{2.2}$$

Problem 3. Let Ω be a bounded domain and $T > 0$ be a given final time. For $f \in C^0([0,T]; L_2(\Omega))$ and $u_0 \in H_0^1(\Omega)$ given, we consider the parabolic problem consisting in finding $u : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) - \Delta u(x,t) = f(x,t) & \text{for } (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

We assume that the solution u to the above problem is sufficiently smooth.

Let N be a strictly positive integer and let $\tau := T/N$, $t_n := n\tau$ and $t^{n+\frac{1}{2}} := \frac{1}{2}(t^{n+1} + t^n)$ for $n = 0, ..., N$. We consider the following semi-discretization in time: Set $U^0 := u_0$ and define $U^n \in H_0^1(\Omega)$ recursively by

$$\begin{cases} \frac{1}{\tau}(U^{n+1}(x) - U^n(x)) - \frac{1}{2}\Delta(U^{n+1}(x) + U^n(x)) = f(x, t^{n+\frac{1}{2}}) & \text{for } x \in \Omega, \\ U^{n+1}(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

(a) (Stability) Show that for $n = 0, ..., N$, U^n satisfies

$$\|U^{n+1}\|_{L_2(\Omega)} \leq \|U^0\|_{L_2(\Omega)} + \tau \sum_{j=0}^{n} \|f(t^{j+\frac{1}{2}})\|_{L_2(\Omega)}. \tag{3.1}$$

<u>Hint:</u> Write the time-discretized problem in weak form before multiplying by a suitable test function.

(b) (Consistency I) Show either (but not both) that

$$\|\frac{1}{\tau}(u(t^{n+1}) - u(t^n)) - \frac{\partial}{\partial t}u(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \leq C\tau\|\frac{\partial^3}{\partial t^3}u\|_{L_1(t^n,t^{n+1};L_2(\Omega))}$$

or

$$\|\frac{1}{2}\Delta\left(u(t^{n+1}) + u(t^n)\right) - \Delta u(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \leq C\tau\|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_1(t^n,t^{n+1};L_2(\Omega))}.$$

Here C is a constant independent of τ, T and u.
<u>Hint:</u> You can use without proof the following Taylor expansion formula

$$g(b) = g(a) + g'(a)(b-a) + ... + \frac{1}{n!}g^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b (b-t)^n g^{(n+1)}(t)dt.$$

(c) (Consistency II) Deduce from the previous item that for a constant C independent of τ, T and u we have

$$\begin{aligned} &\|\frac{1}{\tau}(u^{n+1}(x) - u^n(x)) - \frac{1}{2}\Delta(u^{n+1}(x) + u^n(x)) - f(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \\ &\leq C\tau\left(\|\frac{\partial^3}{\partial t^3}u\|_{L_1(t^n,t^{n+1};L_2(\Omega))} + \|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_1(t^n,t^{n+1};L_2(\Omega))}\right). \end{aligned} \tag{3.2}$$

(d) From (3.1) and (3.2), conclude the following estimate for the error $e^n := u(t^n) - U^n$:

$$\|e^N\|_{L_2(\Omega)} \leq C\tau^2\left(\|\frac{\partial^3}{\partial t^3}u\|_{L_1(0,T;L_2(\Omega))} + \|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_1(0,T;L_2(\Omega))}\right),$$

where C is a constant independent of τ, T and u.