Numerical Analysis Part

January 11, 2021

Problem 1. Let $\widehat{K} \subset \mathbb{R}^2$ be the reference triangle with vertices $\widehat{\boldsymbol{z}}_1 = (0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0)$, and $\widehat{\boldsymbol{z}}_3 = (0,1)$. Let $\widehat{S} \subset \mathbb{R}$ be the reference edge with end points $\widehat{\boldsymbol{a}}_1 = 0$ and $\widehat{\boldsymbol{a}}_2 = 1$. Let $k \in \mathbb{N}$. Let $\mathbb{P}_{k,d}$ denote the real vector space composed of the d-variate polynomials of degree at most k.

1. Let $\widehat{\mu}_1$, $\widehat{\mu}_2 \in \mathbb{P}_{1,1}$ be the barycentric coordinates associated with the vertices $\widehat{\boldsymbol{a}}_1$, $\widehat{\boldsymbol{a}}_2$, respectively. Give the expressions of $\widehat{\mu}_1(\widehat{x})$, $\widehat{\mu}_2(\widehat{x})$, (no proof needed).

2. Let $\widehat{\lambda}_1$, $\widehat{\lambda}_2$, $\widehat{\lambda}_3 \in \mathbb{P}_{1,2}$ be the barycentric coordinates associated with the vertices $\widehat{\boldsymbol{z}}_1$, $\widehat{\boldsymbol{z}}_2$, $\widehat{\boldsymbol{z}}_3$, respectively. Give the expressions of $\widehat{\lambda}_1(\widehat{x},\widehat{y})$, $\widehat{\lambda}_2(\widehat{x},\widehat{y})$, $\widehat{\lambda}_3(\widehat{x},\widehat{y})$ (no proof needed).

3. For all $i \in \{1,2,3\}$, let $\widehat{E}_i$ be the edge of $\widehat{K}$ with the endpoints $\widehat{\boldsymbol{z}}_i$ and $\widehat{\boldsymbol{z}}_{i+1}$, with the convention that $\boldsymbol{z}_4 := \boldsymbol{z}_1$. Give the expression of the unique affine geometric mapping $\boldsymbol{T}_{\widehat{E}_i} : \widehat{S} \to \mathbb{R}$ that maps $\widehat{S}$ to $\widehat{E}_i$ and is such that $\boldsymbol{T}_{\widehat{E}_i}(\widehat{\boldsymbol{a}}_1) = \widehat{\boldsymbol{z}}_i$.

4. For all $i \in \{1,2,3\}$, what is the size of the Jacobian matrix of $\boldsymbol{T}_{\widehat{E}_i} : \widehat{S} \to \mathbb{R}$ (i.e., how many rows and columns)? Compute the Jacobian matrix.

5. Let $K \subset \mathbb{R}^2$ be a triangle with vertices $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, and $\boldsymbol{z}_3$ (all assumed to be distinct). How many affine geometric transformations there are that map $\widehat{K}$ to K?

6. Give the expression of the unique affine geometric mapping $\boldsymbol{T}_K : \widehat{K} \to \mathbb{R}^2$ that maps $\widehat{K}$ to K and is such that $\boldsymbol{T}_K(\widehat{\boldsymbol{z}}_i) = \boldsymbol{z}_i$ for all $i \in \{1,2,3\}$.

7. Let $\widehat{P}_{\widehat{K}} := \{\widehat{q}_{|\widehat{K}}, q \in \mathbb{P}_{2,2}\}$ (i.e., $\widehat{P}$ is composed of the restrictions to $\widehat{K}$ of the two-variate polynomials of degree at most 2). For all $i \in \{1,2,3\}$, let $\widehat{\sigma}_i^{\mathrm{v}} \in \mathcal{L}(\mathbb{P}_{1,2};\mathbb{R})$ be defined by setting $\widehat{\sigma}_i^{\mathrm{v}}(\widehat{p}) := \widehat{p}(\widehat{\boldsymbol{z}}_i)$. Let $\widehat{\sigma}_i^{\mathrm{e}} \in \mathcal{L}(\widehat{P};\mathbb{R})$ be defined by setting $\widehat{\sigma}_i^{\mathrm{e}}(\widehat{p}) := \frac{1}{|\widehat{E}_i|}\int_{\widehat{E}_i} \widehat{p}\,\mathrm{d}l$, where $|\widehat{E}_i|$ is the length of $\widehat{E}_i$. Let $\widehat{\Sigma} := \{\widehat{\sigma}_1^{\mathrm{v}}, \widehat{\sigma}_2^{\mathrm{v}}, \widehat{\sigma}_3^{\mathrm{v}}, \widehat{\sigma}_1^{\mathrm{e}}, \widehat{\sigma}_2^{\mathrm{e}}, \widehat{\sigma}_3^{\mathrm{e}}\}$. Prove that $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ is a unisolvent finite element.

Problem 2. Let $D := (0,1)$. Let $V := \{v \in H^1(D) \mid v(0) = 0\}$ equipped with the inner product $\int_D (u'(x)v'(x) + u(x)v(x))\,\mathrm{d}x$. Accept as a fact that V is a Hilbert space. Let $u_0, u_1 \in \mathbb{R}$ and $f \in C^0(D;\mathbb{R})$. Consider the following two-point boundary value problem:

$$
\begin{aligned}
-u''(x) + u(x) &= f(x), \quad x \in D, \\
u(0) &= u_0, \\
u'(1) + u(1) &= u_1.
\end{aligned} \tag{1}
$$

1. Write a weak formulation of this problem.

2. Prove that $|v(1)| \le \|v'\|_{L^2(D)}$ for all $v \in V$. (*Hint*: Use without proving it that $W := \{v \in C^1(D;\mathbb{R}) \mid v(0) = 0\}$ is dense in V.)

3. Prove that the proposed weak formulation is well-posed. (Prove in details that all the assumptions of the theoretical result you invoke are met.)

Problem 3. Consider the problem stated in (1). The purpose of this problem is to construct a finite difference approximation of (1). Let u be the solution to (1) and assume that u has four continuous derivatives on the closed interval $[0,1]$. Let N be a nonzero natural number. Let $h := \frac{1}{N}$ and $x_i := ih$, for $i = 0,\dots,N$. Let us set $f_i := f(x_i)$ for all $i \in \{0,\dots,N\}$. The finite difference approximation of (1) we consider consists of seeking $(y_i)_{i\in\{0,\dots,N\}} \in \mathbb{R}^{N+1}$ so that

$$
\begin{aligned}
y_0 &= u_0 \\
-\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i &= f_i, \qquad i = 1,\dots,N-1, \\
\frac{y_N - y_{N-1}}{h} + y_N &= u_1.
\end{aligned}
$$

1. Let $i \in \{1,\dots,N-1\}$. Using Taylor expansions, compute

$$\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} - u''(x_i).$$

2. Using Taylor expansions, compute $\frac{u(x_N)-u(x_{N-1})}{h} - u'(x_N)$.

3. Prove the following a priori estimate:

$$\max_{0\le j\le N} y_j \le \max\{u_0, u_1\} + \max_{1\le j\le N-1} f(x_j).$$

(*Hint:* If $y_i = \max_{0\le j\le N} y_j$, then $y_i - y_{i+1} \ge 0$ and $y_i - y_{i-1} \ge 0$. Notice also that $-y_{i-1} + 2y_i - y_{i+1} = y_i - y_{i-1} + y_i - y_{i+1}$. Distinguish three cases: the maximum is attained at $i = 0$, at $i \in \{1,\dots,N-1\}$, or at $i = N$.)

4. Prove the following a priori estimate:

$$\max_{0\le j\le N} |y_i| \le \max\{|u_0|, |u_1|\} + \max_{1\le j\le N-1} |f(x_i)|.$$

(*Hint:* reason as above to derive an estimate on $\min_{0\le i\le N} y_j$ and conclude.)

5. Introduce the error $e_i := y_i - u(x_i)$ and show that

$$\max_{0\le i\le N} |e_i| \le \frac{h}{2} \max\{\max_{0\le x\le 1} |u''(x)|, \frac{h}{6} \max_{0\le x\le 1} |u^{(4)}(x)|\}.$$