Applied Analysis Qualifying Exam  
May 22, 2007

**Instructions:** Do any 7 of the 9 problems in this exam. Show all of your work clearly. Please indicate which 2 of the 9 problems you are skipping.

**Policy on misprints:** The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. State and prove one of the following theorems:
   
   (a) The Weierstrass approximation theorem (sketch of proof suffices).
   
   (b) The Hilbert space projection (decomposition) theorem.
   
   (c) The Shannon sampling theorem.

2. Suppose that $f(\theta)$ is $2\pi$-periodic function in $C^{(1)}(\mathbb{R})$, and that $f''$ is piecewise continuous and $2\pi$-periodic. Let $c_k$ denote the $k^{th}$ (complex) Fourier coefficient for $f$, and let $f_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the $n^{th}$ partial sum of the Fourier series for $f$, $n \geq 1$.

   (a) For $k \neq 0$, show that the Fourier coefficient $c_k$ satisfies the bound
   
   $$ |c_k| \leq \frac{1}{2\pi |k|^2} \|f''\|_{L^1[0,2\pi]}.$$  

   (b) Show that both of these hold for $f$. (The constants are independent of $f$ and $n$.)

   $$ \|f-f_n\|_{L^2[0,2\pi]} \leq C_1 \frac{\|f''\|_{L^1[0,2\pi]}}{\sqrt{n}} \quad \text{and} \quad \|f-f_n\|_{C[0,2\pi]} \leq C_2 \frac{\|f''\|_{L^1[0,2\pi]}}{n}.$$ 

3. Consider the integral operator $Ku = \int_{a}^{b} k(x, \xi) u(\xi) d\xi$.

   (a) Sketch a proof of this: *If $K$ is a Hilbert-Schmidt operator, then $K$ is compact.*
(b) Let $Ku = \int_0^\pi k(x, \xi)u(\xi)d\xi$, where

$$k(x, \xi) = \begin{cases} 
    x - \pi & 0 \leq \xi \leq x \leq \pi, \\
    \xi - \pi & 0 \leq x < \xi \leq \pi.
\end{cases}$$

Explain why this $K$ is compact. Show that it is self adjoint and find the eigenvalues and eigenfunctions for $K$. (Hint: convert the integral equation into a differential equation plus boundary conditions.)

(c) With $K$ as in part (b), for what values of $\lambda$ will $u = f + \lambda Ku$ have a solution for all $f \in L^2[a, b]$? Why?

4. Let $D$ be the set of compactly supported $C^\infty$ functions defined on $\mathbb{R}$ and let $D'$ be the corresponding set of distributions.

(a) Define convergence in $D$ and in $D'$.

(b) Show that every $\psi \in D$ satisfies $\psi(x) = (x^2 \varphi(x))'$ for some $\varphi \in D$ if and only if

$$\int_{-\infty}^{\infty} \psi(x)dx = \int_0^{\infty} \psi(x)dx = \psi(0) = 0.$$

(c) Use the result above to find all $t \in D'$ that solve $x^2t' = 0$, in the sense of distributions.

5. A mass $m$ is subject to a force due to a radial potential $V = V(r)$, where $r$ is the radius in spherical coordinates. The angles $\theta$ and $\varphi$ are the colatitude and longitude, respectively.

(a) Find the system’s Lagrangian in spherical coordinates.

(b) Find the momenta $p_r$, $p_\theta$ and $p_\varphi$ conjugate to $r$, $\theta$ and $\varphi$, respectively, and also the Hamiltonian $H(r, \theta, \varphi, p_r, p_\theta, p_\varphi)$ for the system.

(c) Write down Hamilton’s equations for the system. Use them to show that $H$ is a constant of the motion.

6. Use Laplace’s method and Watson’s lemma to find the first two terms of an asymptotic expansion for

$$I(x) = \int_0^\infty e^{-x \cosh(t)} \sinh^{1/2}(t)dt, \quad x \to +\infty.$$
7. Let \( \sigma \geq 0 \) and consider the Sturm-Liouville problem \((xu')' + \lambda xu = 0\), with \( u(0) \) bounded and \( u'(1) + \sigma u(1) = 0 \).

(a) Show that this S-L problem has the solution \( u = J_0(\sqrt{\lambda}x) \), where \( J_0 \) is the 0 order Bessel function, and where the eigenvalues must satisfy \( \sigma J_0(\sqrt{\lambda}) + \sqrt{\lambda}J_0'(\sqrt{\lambda}) = 0 \).

(b) Write out the functional that must be minimized by \( u \), subject to the constraint \( H(u) = \int_0^1 u^2(x)xdx = 1 \), to get the S-L problem and the boundary conditions.

(c) Use the Courant-Fischer minimax principle to determine how the \( k^{th} \) eigenvalue \( \lambda_k(\sigma) \) behaves as \( \sigma \) increases from 0.

8. Consider the Schrödinger operator with a \( \delta \)-function potential, \( Hu = -u'' + \alpha \delta(x)u \), where \( \alpha > 0 \). For a plane wave incoming from \(-\infty\), find the reflection and transmission coefficients.

9. Let \( Lu = -x(xu')' \) be defined on functions satisfying the boundary condition that \( u(0) = 0 \), and let \( \mathcal{H} \) be the weighted \( L^2 \) space, with the inner product \( \langle f, g \rangle = \int_0^\infty f(x)g(x)\frac{dx}{x} \). You are given that \( L \) will be self adjoint if its domain is \( \mathcal{D}_L = \{ u \in \mathcal{H} \mid Lu \in \mathcal{H} \text{ and } u(0) = 0 \} \).

(a) Find the Green’s function \( G(x, \xi; z) \) for \(-x(xG')' - zG = \delta(x - \xi)\), with \( G(0, \xi; z) = 0 \), \( G(x, \xi; z) \in L^2[0, \infty) \). (This is the kernel for the resolvent \((L - zI)^{-1}\).)

(b) Employ the spectral theorem (and Stone’s formula) to obtain the Mellin transform formulas,

\[
F(s) = \int_0^\infty x^{s-1}f(x)dx \quad \text{and} \quad f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} x^{-s}F(s)ds .
\]