A Note on Guaranteed Sparse Recovery via ℓ_1-Minimization

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Abstract

It is proved that every s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ can be recovered from the measurement vector $\mathbf{y} = A\mathbf{x} \in \mathbb{C}^m$ via ℓ_1-minimization as soon as the $2s$-th restricted isometry constant of the matrix A is smaller than $3/(4+\sqrt{6}) \approx 0.4652$, or smaller than $4/(6+\sqrt{6}) \approx 0.4734$ for large values of s.

We consider in this note the classical problem of Compressive Sensing consisting in recovering an s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ from the mere knowledge of a measurement vector $\mathbf{y} = A\mathbf{x} \in \mathbb{C}^m$, with $m \ll N$, by solving the minimization problem

$$(\mathrm{P}_1) \qquad \underset{\mathbf{z} \in \mathbb{C}^N}{\text{minimize}} \ \|\mathbf{z}\|_1 \qquad \text{subject to } A\mathbf{z} = \mathbf{y}.$$

A much favored tool in the analysis of (P_1) has been the restricted isometry constants δ_k of the $m \times N$ measurement matrix A, defined as the smallest positive constants δ such that

$$(1) \qquad (1-\delta)\|\mathbf{z}\|_2^2 \le \|A\mathbf{z}\|_2^2 \le (1+\delta)\|\mathbf{z}\|_2^2 \qquad \text{for all } k\text{-sparse vector } \mathbf{z} \in \mathbb{C}^N.$$

This notion was introduced by Candès and Tao in [3], where it was shown that all s-sparse vectors are recovered as unique solutions of (P_1) as soon as $\delta_{3s} + 3\delta_{4s} < 2$. There are many such sufficient conditions involving the constants δ_k, but we find a condition involving only δ_{2s} more natural, since it is known [3] that an algorithm recovering all s-sparse vectors $\mathbf{x}$ from the measurements $\mathbf{y} = A\mathbf{x}$ exists if and only if $\delta_{2s} < 1$. Candès showed in [2] that s-sparse recovery is guaranteed as soon as $\delta_{2s} < \sqrt{2} - 1 \approx 0.4142$. This sufficient condition was later improved to $\delta_{2s} < 2/\big(3+\sqrt{2}\big) \approx 0.4531$ in [5], and to $\delta_{2s} < 2/\big(2+\sqrt{5}\big) \approx 0.4721$ in [1], with the proviso that s is either large or a multiple of 4. The purpose of this note is to show that the threshold on δ_{2s} can be pushed further — we point out that Davies and Gribonval proved that it cannot be pushed further than $1/\sqrt{2} \approx 0.7071$ in [4]. Our proof relies heavily on a technique introduced in [1]. Let us note that the results of [2], [5], and [1], even though stated for $\mathbb{R}$ rather than $\mathbb{C}$, are valid in both settings. Indeed, for disjointly supported vectors $\mathbf{u}$ and $\mathbf{v}$, instead of using a real polarization formula to derive the estimate

$$(2) \qquad |\langle A\mathbf{u}, A\mathbf{v}\rangle| \le \delta_k \|\mathbf{u}\|_2 \|\mathbf{v}\|_2,$$

where k is the size of $\mathrm{supp}(\mathbf{u}) \cup \mathrm{supp}(\mathbf{v})$, we remark that $\delta_k = \max\big\{\|A_K^* A_K - I\|_2, \mathrm{card}(K) \le k\big\}$, so that

$$|\langle A\mathbf{u}, A\mathbf{v}\rangle| = |\langle A_K\mathbf{u}, A_K\mathbf{v}\rangle| = |\langle A_K^* A_K\mathbf{u}, \mathbf{v}\rangle| = |\langle (A_K^* A_K - I)\mathbf{u}, \mathbf{v}\rangle| \le \delta_k \|\mathbf{u}\|_2 \|\mathbf{v}\|_2.$$

Using (2), we can establish our main result in the complex setting, as stated below.

Theorem 1. Every s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ is the unique minimizer of (P_1) with $\mathbf{y} = A\mathbf{x}$ if

$$\delta_{2s} < \frac{3}{4+\sqrt{6}} \approx 0.4652,$$

and, for large s, if

$$\delta_{2s} < \frac{4}{6+\sqrt{6}} \approx 0.4734.$$

This theorem is a consequence of the following two propositions.

Proposition 2. Every s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ is the unique minimizer of (P_1) with $\mathbf{y} = A\mathbf{x}$ if

1) $\delta_{2s} < \frac{1}{2}$ when $s = 1$,

2) $\delta_{2s} < \frac{3}{4+\sqrt{(6s-2r)/(s-1)}}$ when $s = 3n + r$ with $1 \le r \le 3$,

3) $\delta_{2s} < \frac{4}{5+\sqrt{(12s-3r)/(s-1)}}$ when $s = 4n + r$ with $1 \le r \le 4$,

4) $\delta_{2s} < \frac{2}{3+\sqrt{1+s/(8n+\lfloor 8r/5 \rfloor)}}$ when $s = 5n + r$ with $1 \le r \le 5$.

Proposition 3. Every s-sparse vector $\mathbf{x} \in \mathbb{C}^N$ is the unique minimizer of (P_1) with $\mathbf{y} = A\mathbf{x}$ if

$$\delta_{2s} < \frac{1}{1+\sqrt{s\,\widetilde{s}/\big(8(\widetilde{s}-s)(3s-2\widetilde{s})\big)}} \qquad \text{where } \widetilde{s} = \lfloor \sqrt{3/2}\,s \rfloor. \tag{3}$$

Proof of Theorem 1. For $2 \le s \le 8$, we determine which sufficient condition of Proposition 2 is the weakest, using the following table of values for the thresholds

	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	$s=8$
Case 2)	0.4393	**0.4652**	0.4472	0.4580	0.4652	0.4558	0.4610
Case 3)	0.4328	0.4611	**0.4726**	0.4558	0.4633	**0.4686**	**0.4726**
Case 4)	**0.4661**	0.4627	0.4661	**0.4679**	**0.4661**	0.4674	0.4661

For these values of s, requiring $\delta_{2s} < 0.4652$ is enough to guarantee s-sparse recovery. As for the values $s \ge 9$, since the function of n appearing in Case 4) is nondecreasing when r is fixed, the corresponding sufficient condition holds for s as soon as it holds for $s-5$. Then, because requiring $\delta_{2s} < 0.4661$ is enough to guarantee s-sparse recovery from Case 4) when $4 \leq s \leq 8$, it is also enough to guarantee it when $s \geq 9$. Taking Case 1) into account, we conclude that the inequality $\delta_{2s} < 0.4652$ ensures s-sparse recovery for every integer $s \geq 1$, as stated in the first part of Theorem 1. The second part of Theorem 1 follows from Proposition 3 by writing

$$\frac{s\,\widetilde{s}}{8(\widetilde{s}-s)(3s-2\widetilde{s})} \underset{s\to\infty}{\longrightarrow} \frac{\sqrt{3/2}}{8(\sqrt{3/2}-1)(3-2\sqrt{3/2})} = \frac{6}{16(3-\sqrt{6})^2} = \left(\frac{\sqrt{6}}{4(3-\sqrt{6})}\right)^2 = \left(\frac{2+\sqrt{6}}{4}\right)^2,$$

and substituting this limit into (3). □

A crucial role in the proofs of Propositions 2 and 3 is played by the following lemma, which is simply the shifting inequality introduced in [1] when $k \leq 4\ell$. We provide a different proof for the reader's convenience.

Lemma 4. Given integers $k, \ell \geq 1$, for a sequence $a_1 \geq a_2 \geq \cdots \geq a_{k+\ell} \geq 0$, one has

$$\left[\sum_{j=\ell+1}^{\ell+k} a_j^2\right]^{1/2} \leq \max\left[\frac{1}{\sqrt{4\ell}}, \frac{1}{\sqrt{k}}\right]\left[\sum_{j=1}^{k} a_j\right].$$

Proof. The case $\ell+1 \geq k$ follows from the facts that the left-hand side is at most $\sqrt{k}\,a_{\ell+1}$ and that the right-hand side is at least $\sqrt{k}\,a_k$. We now assume that $\ell+1 < k$, so that the subsequences $(a_1,\ldots,a_k)$ and $(a_{\ell+1},\ldots,a_{\ell+k})$ overlap on $(a_{\ell+1},\ldots,a_k)$. Since the left-hand side is maximized when $a_{k+1},\ldots,a_\ell$ all equal a_k, while the right-hand side is minimized when $a_1,\ldots,a_\ell$ all equal $a_{\ell+1}$, it is necessary and sufficient to establish that

$$\left[a_{\ell+1}^2 + \cdots + a_{k-1}^2 + (\ell+1)a_k^2\right]^{1/2} \leq \max\left[\frac{1}{\sqrt{4\ell}}, \frac{1}{\sqrt{k}}\right]\left[(\ell+1)a_{\ell+1} + a_{\ell+2} + \cdots + a_k\right].$$

By homogeneity, this is the problem of maximization of the convex function

$$f(a_{\ell+1},\ldots,a_k) := \left[a_{\ell+1}^2 + \cdots + a_{k-1}^2 + (\ell+1)a_k^2\right]^{1/2}$$

over the convex polygon

$$\mathcal{P} := \left\{(a_{\ell+1},\ldots,a_k) \in \mathbb{R}^{k-\ell} : a_{\ell+1} \geq \cdots \geq a_k \geq 0 \text{ and } (\ell+1)a_{\ell+1} + a_{\ell+2} + \cdots + a_k \leq 1\right\}.$$

Because any point in $\mathcal{P}$ is a convex combination of its vertices and because the function f is convex, its maximum over $\mathcal{P}$ is attained at a vertex of $\mathcal{P}$. We note that the vertices of $\mathcal{P}$ are obtained as intersections of $(k-\ell)$ hyperplanes arising by turning $(k-\ell)$ of the $(k-\ell+1)$ inequality constraints into equalities. We have the following possibilities:

- if $a_{\ell+1} = \cdots = a_k = 0$, then $f(a_{\ell+1}, \ldots, a_k) = 0$;
- if $a_{\ell+1} = \cdots = a_j < a_{j+1} = \cdots = a_k = 0$ and $(\ell+1)a_{\ell+1} + a_{\ell+2} + \cdots + a_k = 1$ for $\ell+1 \leq j \leq k$, then $a_{\ell+1} = \cdots = a_j = 1/j$, so that $f(a_{\ell+1}, \ldots, a_k) = \big[(j-\ell)/j^2\big]^{1/2} \leq \big[1/(4\ell)\big]^{1/2}$ when $j < k$ and that $f(a_{\ell+1}, \ldots, a_k) = \big[k/k^2\big]^{1/2} = \big[1/k\big]^{1/2}$ when $j = k$.

It follows that the maximum of the function f over the convex polygon $\mathcal{P}$ does not exceed $\max\left[1/\sqrt{4\ell}, 1/\sqrt{k}\right]$, which is the expected result. □

Proof of Proposition 2. It is well-known, see e.g. [6], that the recovery of s-sparse vectors is equivalent to the null space property, which asserts that, for any nonzero vector $\mathbf{v} \in \ker A$ and any index set S of size s, one has

$$\|\mathbf{v}_S\|_1 < \|\mathbf{v}_{\overline{S}}\|_1. \tag{4}$$

The notation $\overline{S}$ stands for the complementary of S in $\{1, \ldots, N\}$. Let us now fix a nonzero vector $\mathbf{v} \in \ker A$. We may assume without loss of generality that the entries of $\mathbf{v}$ are sorted in decreasing order

$$|v_1| \geq |v_2| \geq \cdots \geq |v_N|.$$

It is then necessary and sufficient to establish (4) for the set $S = \{1, \ldots, s\}$.

We start by examining Case 4). We partition $\overline{S} = \{s+1, \ldots, N\}$ in two ways as $\overline{S} = S' \cup T_1 \cup T_2 \cup \ldots$ and as $\overline{S} = S' \cup U_1 \cup U_2 \cup \ldots$, where

$$\begin{array}{lll} S' := \{s+1, \ldots, s+s'\} & & \text{is of size } s', \\ T_1 := \{s+s'+1, \ldots, s+s'+t\}, & T_2 := \{s+s'+t+1, \ldots, s+s'+2t\}, \ldots & \text{are of size } t, \\ U_1 := \{s+s'+1, \ldots, s+s'+u\}, & U_2 := \{s+s'+u+1, \ldots, s+s'+2u\}, \ldots & \text{are of size } u. \end{array}$$

We impose the sizes of the sets $S \cup S'$, $S \cup T_k$, and $S' \cup U_k$ to be at most $2s$, i.e.

$$s' \leq s, \qquad t \leq s, \qquad s' + u \leq 2s.$$

Thus, with $\delta := \delta_{2s}$, we derive from (1) and (2)

$$\begin{aligned} \|\mathbf{v}_S + \mathbf{v}_{S'}\|_2^2 &\leq \frac{1}{1-\delta}\|A(\mathbf{v}_S + \mathbf{v}_{S'})\|_2^2 = \frac{1}{1-\delta}\Big[\langle A(\mathbf{v}_S), A(\mathbf{v}_S + \mathbf{v}_{S'})\rangle + \langle A(\mathbf{v}_{S'}), A(\mathbf{v}_S + \mathbf{v}_{S'})\rangle\Big] \\ &= \frac{1}{1-\delta}\Big[\langle A(\mathbf{v}_S), \sum_{k\geq 1} A(-\mathbf{v}_{T_k})\rangle + \langle A(\mathbf{v}_{S'}), \sum_{k\geq 1} A(-\mathbf{v}_{U_k})\rangle\Big] \\ &\leq \frac{1}{1-\delta}\Big[\delta\|\mathbf{v}_S\|_2 \sum_{k\geq 1}\|\mathbf{v}_{T_k}\|_2 + \delta\|\mathbf{v}_{S'}\|_2 \sum_{k\geq 1}\|\mathbf{v}_{U_k}\|_2\Big]. \end{aligned} \tag{5}$$

Introducing the shifted sets $\widetilde{T}_1 := \{s+1, \ldots, s+t\}$, $\widetilde{T}_2 := \{s+t+1, \ldots, s+2t\}$, ..., and $\widetilde{U}_1 := \{s+1, \ldots, s+u\}$, $\widetilde{U}_2 := \{s+u+1, \ldots, s+2u\}$, ..., Lemma 4 yields, for $k \geq 1$,

$$\|\mathbf{v}_{T_k}\|_2 \leq \max\left[\frac{1}{\sqrt{4s'}}, \frac{1}{\sqrt{t}}\right] \|\mathbf{v}_{\widetilde{T}_k}\|_1, \qquad \|\mathbf{v}_{U_k}\|_2 \leq \max\left[\frac{1}{\sqrt{4s'}}, \frac{1}{\sqrt{u}}\right] \|\mathbf{v}_{\widetilde{U}_k}\|_1.$$

Substituting into (5), we obtain

$$\text{(6)} \qquad \|\mathbf{v}_S + \mathbf{v}_{S'}\|_2^2 \leq \frac{\delta}{1-\delta} \left[\|\mathbf{v}_S\|_2 \max\left[\frac{1}{\sqrt{4s'}}, \frac{1}{\sqrt{t}}\right] \|\mathbf{v}_{\overline{S}}\|_1 + \|\mathbf{v}_{S'}\|_2 \max\left[\frac{1}{\sqrt{4s'}}, \frac{1}{\sqrt{u}}\right] \|\mathbf{v}_{\overline{S}}\|_1 \right].$$

To minimize the first maximum in (6), we have all interest in taking the free variable t as large as possible, i.e. $t = s$. We now concentrate on the second maximum in (6). The point (s', u) belongs to the region

$$\mathcal{R} := \big\{ s' \geq 0, u \geq 0, s' \leq s, s' + u \leq 2s \big\}.$$

This region is divided in two by the line $\mathcal{L}$ of equation $u = 4s'$. Below this line, the maximum equals $1/\sqrt{u}$, which is minimized for a large u. Above this line, the maximum equals $1/\sqrt{4s'}$, which is minimized for large s'. Thus, the maximum is minimized at the intersection of the line $\mathcal{L}$ with the boundary of the region $\mathcal{R}$ — other than the origin — which is given by

$$s'_* := \frac{2s}{5}, \qquad u_* := \frac{8s}{5}.$$

If s is a multiple of 5, we can choose (s', u) to be (s'_*, u_*). In view of $4s'_* \geq s$, (6) becomes

$$\|\mathbf{v}_S\|_2^2 + \|\mathbf{v}_{S'}\|_2^2 \leq \frac{\delta}{1-\delta} \frac{\|\mathbf{v}_{\overline{S}}\|_1}{\sqrt{s}} \left[\|\mathbf{v}_S\|_2 + \sqrt{c}\, \|\mathbf{v}_{S'}\|_2 \right] \qquad \text{with } c = \frac{5}{8}.$$

Completing the squares, we obtain, with $\gamma := \big(\delta \|\mathbf{v}_{\overline{S}}\|_1\big) / \big(2(1-\delta)\sqrt{s}\big)$,

$$\big(\|\mathbf{v}_S\|_2 - \gamma\big)^2 + \big(\|\mathbf{v}_{S'}\|_2 - \sqrt{c}\,\gamma\big)^2 \leq (1+c)\gamma^2.$$

Simply using the inequality $\big(\|\mathbf{v}_{S'}\|_2 - \sqrt{c}\,\gamma\big)^2 \geq 0$, we deduce

$$\|\mathbf{v}_S\|_2 \leq \big(1 + \sqrt{1+c}\big)\gamma.$$

Finally, in view of $\|\mathbf{v}_S\|_1 \leq \sqrt{s}\|\mathbf{v}_S\|_2$, we conclude

$$\|\mathbf{v}_S\|_1 \leq \frac{\big(1+\sqrt{1+c}\big)\delta}{2\big(1-\delta\big)} \|\mathbf{v}_{\overline{S}}\|_1.$$

Thus, the null space property (4) is satisfied as soon as

$$\big(1+\sqrt{1+c}\big)\delta < 2\big(1-\delta\big), \qquad \text{i.e.} \qquad \delta < \frac{2}{3+\sqrt{1+c}}.$$

Substituting $c = 5/8$ leads to the sufficient condition $\delta_{2s} < 2/\big(3 + \sqrt{13/8}\big) \approx 0.4679$, valid when s is a multiple of 5. When s is not a multiple of 5, we cannot choose (s', u) to be (s'_*, u_*), and we choose it to be a corner of the square $\big[\lfloor s'_* \rfloor, \lceil s'_* \rceil\big] \times \big[\lfloor u_* \rfloor, \lceil u_* \rceil\big]$. In all cases, the corner $\big(\lceil s'_* \rceil, \lceil u_* \rceil\big)$ is inadmissible since $\lceil s'_* \rceil + \lceil u_* \rceil > 2s$, and among the three admissible corners, one can verify that the smallest value of $\max\big[1/\sqrt{4s'}, 1/\sqrt{u}\big]$ is achieved for $\big(s', u\big) = \big(\lceil s'_* \rceil, \lfloor u_* \rfloor\big)$. With this choice, in view of $4s' \geq s$, (6) becomes

$$\|\mathbf{v}_S\|_2^2 + \|\mathbf{v}_{S'}\|_2^2 \leq \frac{\delta}{1-\delta} \frac{\|\mathbf{v}_{\overline{S}}\|_1}{\sqrt{s}} \Big[\|\mathbf{v}_S\|_2 + \sqrt{c}\, \|\mathbf{v}_{S'}\|_2\Big] \qquad \text{with } c = \frac{s}{\lfloor 8s/5 \rfloor}.$$

The same arguments as before yield the sufficient condition $\delta_{2s} < 2/\big(3 + \sqrt{1 + s/\lfloor 8s/5 \rfloor}\big)$, which is nothing else than Condition 4).

We now turn to Cases 2) and 3), which we treat simultaneously by writing $s = pn + r$, $1 \leq r \leq p$, for $p = 3$ and $p = 4$. We partition $\overline{S}$ as $\overline{S} = S' \cup T_1 \cup T_2 \cup \ldots$ and $\overline{S} = S' \cup U_1 \cup U_2 \cup \ldots$, where

$S' := \{s+1, \ldots, s+s'\}$	is of size $s' = n+1$,
$T_1 := \{s+s'+1, \ldots, s+s'+t\},\ T_2 := \{s+s'+t+1, \ldots, s+s'+2t\}, \ldots$	are of size $t = s$,
$U_1 := \{s+s'+1, \ldots, s+s'+u\}, U_2 := \{s+s'+u+1, \ldots, s+s'+2u\}, \ldots$	are of size $u = s-1$.

Moreover, we partition S as $S_1 \cup \cdots \cup S_p$, where $S_1, \ldots, S_r$ are of size $n+1$ and $S_{r+1}, \ldots, S_p$ of size n. We then set $\mathbf{w}_0 := \mathbf{v}_S$, $\mathbf{w}_1 := \mathbf{v}_{S_2} + \cdots + \mathbf{v}_{S_p} + \mathbf{v}_{S'}$, $\mathbf{w}_2 := \mathbf{v}_{S_1} + \mathbf{v}_{S_3} + \cdots + \mathbf{v}_{S_p} + \mathbf{v}_{S'}$, $\ldots$, $\mathbf{w}_p := \mathbf{v}_{S_1} + \ldots + \mathbf{v}_{S_{p-1}} + \mathbf{v}_{S'}$, so that

$$\sum_{j=0}^{p} \mathbf{w}_j = p(\mathbf{v}_S + \mathbf{v}'_S) \qquad \text{and} \qquad \sum_{j=0}^{p} \|\mathbf{w}_j\|_2^2 = p\|\mathbf{v}_S + \mathbf{v}'_S\|_2^2.$$

With $\delta := \delta_{2s}$, we derive from (1) and (2)

$$\begin{aligned}
\|\mathbf{v}_S + \mathbf{v}'_S\|_2^2 &\leq \frac{1}{1-\delta}\|A(\mathbf{v}_S + \mathbf{v}'_S)\|_2^2 = \frac{1}{1-\delta}\big\langle A\big(\sum_{j=0}^{p} \mathbf{w}_j/p\big), A\big(-\mathbf{v}_{\overline{S \cup S'}}\big)\big\rangle \\
&= \frac{1}{1-\delta}\,\frac{1}{p}\Big[\sum_{j=0}^{r} \big\langle A(\mathbf{w}_j), \sum_{k \geq 1} A(-\mathbf{v}_{T_k})\big\rangle + \sum_{j=r+1}^{p} \big\langle A(\mathbf{w}_j), \sum_{k \geq 1} A(-\mathbf{v}_{U_k})\big\rangle\Big] \\
&\leq \frac{1}{1-\delta}\,\frac{1}{p}\Big[\sum_{j=0}^{r} \delta\|\mathbf{w}_j\|_2 \sum_{k \geq 1} \|\mathbf{v}_{T_k}\|_2 + \sum_{j=r+1}^{p} \delta\|\mathbf{w}_j\|_2 \sum_{k \geq 1} \|\mathbf{v}_{U_k}\|_2\Big].
\end{aligned} \tag{7}$$

Taking into account that $s \leq 4s'$ and $s - 1 \leq 4s'$, Lemma 4 yields, for $k \geq 1$,

$$\|\mathbf{v}_{T_k}\|_2 \leq \frac{1}{\sqrt{s}}\|\mathbf{v}_{\widetilde{T}_k}\|_1, \qquad \|\mathbf{v}_{U_k}\|_2 \leq \frac{1}{\sqrt{s-1}}\|\mathbf{v}_{\widetilde{U}_k}\|_1,$$

where $\widetilde{T}_1 := \{s+1, \ldots, s+t\}$, $\widetilde{T}_2 := \{s+t+1, \ldots, s+2t\}$, $\ldots$, and $\widetilde{U}_1 := \{s+1, \ldots, s+u\}$,

$\widetilde{U}_2 := \{s+u+1, \ldots, s+2u\}, \ldots$. Substituting into (7), we obtain

$$p\|\mathbf{v}_S + \mathbf{v}_{S'}\|_2^2 \;\leq\; \frac{1}{1-\delta}\,\frac{1}{p}\Bigg[\sum_{j=0}^{r} \delta \|\mathbf{w}_j\|_2 \frac{\|\mathbf{v}_{\overline{S}}\|_1}{\sqrt{s}} + \sum_{j=r+1}^{p} \delta \|\mathbf{w}_j\|_2 \frac{\|\mathbf{v}_{\overline{S}}\|_1}{\sqrt{s-1}}\Bigg]$$

(8)
$$= \frac{\delta}{1-\delta}\,\frac{\|\mathbf{v}_{\overline{S}}\|_1}{\sqrt{s}}\Bigg[\|\mathbf{v}_S\|_2 + \sum_{j=1}^{r}\|\mathbf{w}_j\|_2 + \sum_{j=r+1}^{p}\sqrt{\frac{s}{s-1}}\|\mathbf{w}_j\|_2\Bigg].$$

We use the Cauchy–Schwarz inequality to derive

$$\sum_{j=1}^{r}\|\mathbf{w}_j\|_2 + \sum_{j=r+1}^{p}\sqrt{\frac{s}{s-1}}\|\mathbf{w}_j\|_2 \;\leq\; \sqrt{r + (p-r)\frac{s}{s-1}}\,\sqrt{\sum_{j=1}^{r}\|\mathbf{w}_j\|_2^2}$$

(9)
$$= \sqrt{\frac{ps-r}{s-1}}\,\sqrt{p\|\mathbf{v}_S + \mathbf{v}_{S'}\|_2^2 - \|\mathbf{v}_S\|_2^2}.$$

Setting $a := \|\mathbf{v}_S\|_2$ and $b := \sqrt{p\|\mathbf{v}_S + \mathbf{v}_{S'}\|_2^2 - \|\mathbf{v}_S\|_2^2}$, (8) and (9) imply

$$a^2 + b^2 \leq \frac{\delta}{1-\delta}\,\frac{\|\mathbf{v}_{\overline{S}}\|_1}{\sqrt{s}}\Big[a + \sqrt{c}\,b\Big], \qquad \text{with } c := \frac{ps-r}{s-1}.$$

Completing the squares, we obtain, with $\gamma := \big(\delta\|\mathbf{v}_{\overline{S}}\|_1\big)/\big(2(1-\delta)\sqrt{s}\big)$,

$$(a-\gamma)^2 + (b - \sqrt{c}\,\gamma)^2 \leq (1+c)\gamma^2.$$

Thus, the point (a,b) is inside the circle $\mathcal{C}$ passing through the origin, with center $(\gamma, \sqrt{c}\,\gamma)$. Since $b \geq \sqrt{p-1}\,a$, this point is above the line $\mathcal{L}$ passing through the origin, with slope $\sqrt{p-1}$. If (a_*, b_*) denotes the intersection of $\mathcal{C}$ and $\mathcal{L}$ — other than the origin — we then have

$$a \leq a_* = \frac{2\big(1 + \sqrt{c\,(p-1)}\big)}{p}\,\gamma,$$

as one can verify that the point on the circle $\mathcal{C}$ with maximal abscissa is below the line $\mathcal{L}$. Finally, in view of $\|\mathbf{v}_S\|_1 \leq \sqrt{s}\|\mathbf{v}_S\|_2 = \sqrt{s}\,a$, we conclude that

$$\|\mathbf{v}_S\|_1 \leq \frac{1 + \sqrt{c\,(p-1)}}{p}\,\frac{\delta}{1-\delta}\,\|\mathbf{v}_{\overline{S}}\|_1.$$

Thus, the null space property (4) is satisfied as soon as

$$\big(1 + \sqrt{c\,(p-1)}\big)\,\delta < p\,\big(1-\delta\big), \qquad \text{i.e.} \qquad \delta < \frac{p}{p + 1 + \sqrt{c\,(p-1)}}.$$

Specifying $p=3$ and $p=4$ yields Conditions 2) and 3), respectively.

As for Case 1), corresponding to $s=1$, we simply write, with $\delta := \delta_2$,

$$\|\mathbf{v}_{\{1\}}\|_2^2 \leq \frac{1}{1-\delta}\|A\mathbf{v}_{\{1\}}\|_2^2 = \frac{1}{1-\delta}\Big\langle A\mathbf{v}_{\{1\}}, \sum_{k\geq 2} A(-\mathbf{v}_{\{k\}})\Big\rangle \leq \frac{\delta}{1-\delta}\|\mathbf{v}_{\{1\}}\|_2 \sum_{k\geq 2}\|\mathbf{v}_{\{k\}}\|_2,$$

so that

$$\|\mathbf{v}_{\{1\}}\|_1 = \|\mathbf{v}_{\{1\}}\|_2 \leq \frac{\delta}{1-\delta}\|\mathbf{v}_{\overline{\{1\}}}\|_1,$$

and the null space property (4) is satisfied as soon as $\delta/(1-\delta) < 1$, i.e. $\delta < 1/2$. □

Proof of Proposition 3. As in the previous proof, we only need to establish that $\|\mathbf{v}_S\|_1 < \|\mathbf{v}_{\overline{S}}\|_1$ for a nonzero vector $\mathbf{v} \in \ker A$ sorted with $|v_1| \geq \cdots \geq |v_N|$ and for $S = \{1, \ldots, s\}$. We partition $\overline{S} = \{s+1, \ldots, N\}$ as $\overline{S} = S' \cup T_1 \cup T_2 \cup \ldots$, where

$$\begin{array}{ll} S' := \{s+1, \ldots, s+s'\} & \text{is of size } s', \\ T_1 := \{s+s'+1, \ldots, s+s'+t\}, \quad T_2 := \{s+s'+t+1, \ldots, s+s'+2t\}, \ldots & \text{are of size } t. \end{array}$$

For an integer $r \leq s+s'$, we consider the r-sparse vectors $\mathbf{w}_1 := \mathbf{v}_{\{1,\ldots,r\}}$, $\mathbf{w}_2 := \mathbf{v}_{\{2,\ldots,r+1\}}, \ldots$, $\mathbf{w}_{s+s'-r+1} := \mathbf{v}_{\{s+s'-r+1,\ldots,s+s'\}}$, $\mathbf{w}_{s+s'-r+2} := \mathbf{v}_{\{s+s'-r+2,\ldots,s+s',1\}}, \ldots$, $\mathbf{w}_{s+s'} := \mathbf{v}_{\{s+s',1,\ldots,r-1\}}$, so that

$$\sum_{j=1}^{s+s'} \mathbf{w}_j = r(\mathbf{v}_S + \mathbf{v}_{S'}) \qquad \text{and} \qquad \sum_{j=1}^{s+s'} \|\mathbf{w}_j\|_2^2 = r\|\mathbf{v}_S + \mathbf{v}_{S'}\|_2^2.$$

We impose

$$s' \leq s, \qquad r+t \leq 2s, \qquad t \leq 4s',$$

in order to justify the chain of inequalities, where $\delta := \delta_{2s}$,

$$\begin{aligned} \|\mathbf{v}_S + \mathbf{v}_{S'}\|_2^2 &\leq \frac{1}{1-\delta}\|A(\mathbf{v}_S + \mathbf{v}_{S'})\|_2^2 = \frac{1}{1-\delta}\Big\langle A\Big(\sum_{j=0}^{s+s'} \mathbf{w}_j/r\Big), A\Big(-\sum_{k\geq 1} \mathbf{v}_{T_k}\Big)\Big\rangle \\ &= \frac{1}{1-\delta}\,\frac{1}{r}\sum_{j=1}^{s+s'} \Big\langle A(\mathbf{w}_j), \sum_{k\geq 1} A(-\mathbf{v}_{T_k})\Big\rangle \leq \frac{1}{1-\delta}\,\frac{1}{r}\sum_{j=1}^{s+s'} \delta\|\mathbf{w}_j\|_2 \sum_{k\geq 1}\|\mathbf{v}_{T_k}\|_2 \\ &\leq \frac{\delta}{1-\delta}\,\frac{1}{r}\sum_{j=1}^{s+s'} \|\mathbf{w}_j\|_2\,\frac{\|\mathbf{v}_{\overline{S}}\|_1}{\sqrt{t}} \leq \frac{\delta}{1-\delta}\,\frac{1}{r}\,\sqrt{s+s'}\,\sqrt{\sum_{j=1}^{s+s'}\|\mathbf{w}_j\|_2^2}\,\frac{\|\mathbf{v}_{\overline{S}}\|_1}{\sqrt{t}} \\ &= \sqrt{\frac{s+s'}{rt}}\,\frac{\delta}{1-\delta}\,\|\mathbf{v}_S + \mathbf{v}_{S'}\|_2\,\|\mathbf{v}_{\overline{S}}\|_1. \end{aligned}$$

Simplifying by $\|\mathbf{v}_S + \mathbf{v}_{S'}\|_2$ and using $\|\mathbf{v}_S\|_1 \leq \sqrt{s}\|\mathbf{v}_S\|_1 \leq \sqrt{s}\|\mathbf{v}_S + \mathbf{v}_{S'}\|_2$, we arrive at

$$\begin{aligned} \text{(10)} \qquad \|\mathbf{v}_S\|_1 &\leq \sqrt{\frac{s(s+s')}{rt}}\,\frac{\delta}{1-\delta}\,\|\mathbf{v}_{\overline{S}}\|_1 \\ &= \sqrt{\frac{1+\sigma}{\rho\tau}}\,\frac{\delta}{1-\delta}\,\|\mathbf{v}_{\overline{S}}\|_1, \qquad \text{where } \sigma := \frac{s'}{s},\ \rho := \frac{r}{s},\ \text{and } \tau := \frac{t}{s}. \end{aligned}$$

Pretending that the quantities σ, ρ, τ are continuous variables, we first minimize $(1+\sigma)/(\rho\tau)$, subject to $\sigma \leq 1$, $\rho+\tau \leq 2$, and $\tau \leq 4\sigma$. The minimum is achieved when ρ is largest possible, i.e. $\rho = 2-\tau$. Subsequently, the minimun of $(1+\sigma)/((2-\tau)\tau)$, subject to $\sigma \leq 1$, $\tau \leq 2$, and $\tau \leq 4\sigma$, is achieved when σ is largest possible, i.e. $\sigma = \tau/4$. Finally, one can easily verify that the minimum of $(1+\tau/4)/((2-\tau)\tau)$ subject to $\tau \leq 2$ is achieved for $\tau = 2\sqrt{6}-4$. This corresponds to $\sigma = \sqrt{3/2}-1 \approx 0.2247$, and suggests the choice $s' = (\sqrt{3/2}-1)s$. The latter does not give an integer value for s', so we take $s' = \lfloor (\sqrt{3/2}-1)s \rfloor = \widetilde{s}-s$, and in turn $t = 4s' = 4\widetilde{s}-4s$ and $r = 2s - t = 6s - 4\widetilde{s}$. Substituting into (10), we obtain

$$\|\mathbf{v}_S\|_1 \leq \sqrt{\frac{s\,\widetilde{s}}{8(\widetilde{s}-s)(3s-2\widetilde{s})}}\,\frac{\delta}{1-\delta}\,\|\mathbf{v}_{\overline{S}}\|_1.$$

Thus, the null space property (4) is satisfied as soon as Condition (3) holds. □

Remark. For simplicity, we only considered exactly sparse vectors measured with infinite precision. Standard arguments in Compressive Sensing would show that the same sufficient conditions guarantee a reconstruction that is stable with respect to sparsity defect and robust with respect to measurement error.

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