On Maximal Relative Projection Constants

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Abstract

This article focuses on the maximum of relative projection constants over all m-dimensional subspaces of the N-dimensional coordinate space equipped with the max-norm. This quantity, called maximal relative projection constant, is studied in parallel with a lower bound, dubbed quasimaximal relative projection constant. Exploiting alternative expressions for these quantities, we show how they can be computed when N is small and how to reverse the Kadec–Snobar inequality when N does not tend to infinity. Precisely, we first prove that the (quasi)maximal relative projection constant can be lower-bounded by $c\sqrt{m}$, with c arbitrarily close to one, when N is superlinear in m. The main ingredient is a connection with equiangular tight frames. By using the semicircle law, we then prove that the lower bound $c\sqrt{m}$ holds with $c < 1$ when N is linear in m.

Key words and phrases: projection constants, Seidel matrices, tight frames, equiangular lines, graphs, semicircle law.

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1 Introduction

This article investigates relative projection constants of m-dimensional subspaces $\mathcal{V}_m$ of ℓ_N^∞, which are defined as

$$(1) \qquad \lambda(\mathcal{V}_m, \ell_\infty^N) := \min\left\{ \|P\|_{\infty\to\infty} : \ P \text{ is a projection from } \ell_\infty^N \text{ onto } \mathcal{V}_m \right\},$$

and more specifically maximal relative projection constants, which are defined as

$$(2) \qquad \lambda(m, N) := \max\left\{ \lambda(\mathcal{V}_m, \ell_\infty^N) : \ \mathcal{V}_m \text{ is an } m\text{-dimensional subspace of } \ell_\infty^N \right\}.$$

With $\mathbb{K}$ denoting either $\mathbb{R}$ or $\mathbb{C}$, we append a subscript $\mathbb{K}$ in the notation $\lambda_{\mathbb{K}}(m, N)$ to indicate that $\ell_\infty^N = (\mathbb{K}^N, \|\cdot\|_\infty)$ is understood as a real or a complex linear space. The existing literature often deals with maximal absolute projection constants, which may be defined as

$$\lambda_{\mathbb{K}}(m) := \sup_{N \geq m} \lambda_{\mathbb{K}}(m, N).$$

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As a representative example, the Kadec–Snobar estimate $\lambda_{\mathbb{K}}(m) \leq \sqrt{m}$ can be proved by several different approaches — we can add yet another approach based on Theorem 1, see Remark 10. Following earlier works such as [12] and [5], we focus on the properties of $\lambda_{\mathbb{K}}(m,N)$ with N fixed rather than the properties of $\lambda_{\mathbb{K}}(m)$. In particular, we are looking at reversing the Kadec–Snobar inequality with N moderately large. In Section 2, we highlight two alternative expressions for the maximal relative projection constant $\lambda_{\mathbb{K}}(m,N)$. They are only new in the case $\mathbb{K}=\mathbb{C}$, but even in the case $\mathbb{K}=\mathbb{R}$, the arguments we propose contrast with the ones found in the literature. We also introduce a related quantity $\mu_{\mathbb{K}}(m,N)$, dubbed quasimaximal relative projection constant, which is a lower bound for $\lambda_{\mathbb{K}}(m,N)$. We then establish some common properties shared by $\lambda_{\mathbb{K}}(m,N)$ and $\mu_{\mathbb{K}}(m,N)$. In Section 3, we focus on the computation of these quantities. We show how to determine (for small N) the exact value of $\mu_{\mathbb{R}}(m,N)$ and the value of a lower bound for $\lambda_{\mathbb{R}}(m,N)$, which is in fact believed to be the true value. In particular, we reveal that $\lambda_{\mathbb{R}}(m,N)$ and $\mu_{\mathbb{R}}(m,N)$ really do differ in general. In Section 4, we make explicit a connection between equiangular tight frames and specific values for $\lambda_{\mathbb{K}}(m,N)$ and $\mu_{\mathbb{K}}(m,N)$. Based on these considerations, we prove that the Kadec–Snobar estimate is optimal in the sense that there are spaces $\mathcal{V}_m$ of arbitrarily large dimension m such that $\lambda_{\mathbb{K}}(m)/\sqrt{m}$ (or in fact $\mu_{\mathbb{K}}(m)/\sqrt{m}$) is arbitrarily close to one. This is only new in the case $\mathbb{K}=\mathbb{R}$. However, in the examples provided in Section 4, the dimension N of the superspace grows superlinearly in m. To the best of our knowledge, such a result was previously achieved only with N growing quadratically in m. In Section 5, we further show that a lower estimate $\lambda_{\mathbb{K}}(m,N) \geq c\sqrt{m}$ (or in fact $\mu_{\mathbb{K}}(m,N) \geq c\sqrt{m}$) is actually possible with N growing only linearly in m. For this, we rely on the alternative expression for $\lambda_{\mathbb{K}}(m,N)$ in terms of eigenvalues of Seidel matrices and invoke the semicircle law for such matrices chosen at random. We conclude the article with some remarks linking minimal projections to matrix theory and graph theory via the alternative expression for the maximal relative projection constant $\lambda_{\mathbb{K}}(m,N)$ highlighted at the beginning. Four appendices collect some material whose inclusion in the main text would have disrupted the flow of reading.

Notation: The blackboard-bold letter $\mathbb{K}$ represents either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$. The notation $\mathbb{R}^n_+$ stands for the set of vectors with n nonnegative real entries, just like $\mathbb{K}^n$ stands for the set of vectors with n entries in $\mathbb{K}$. As a linear space, the latter may be equipped with the usual p-norm $\|\cdot\|_p$ for any $p \in [1,\infty]$, in which case it is represented by ℓ_p^n. Given a vector $v \in \mathbb{K}^n$, the notation $\mathrm{diag}(v)$ refers to the diagonal matrix in $\mathbb{K}^{n\times n}$ with v on its diagonal. The modulus (or absolute value) $|M|$ of a matrix $M \in \mathbb{K}^{n\times n}$ is understood componentwise, so that its (i,j)th entry is $|M|_{i,j} = |M_{i,j}|$. The adjoint of a matrix $M \in \mathbb{K}^{n\times n}$ is the matrix M^* with (i,j)th entry $M^*_{i,j} = \overline{M_{j,i}}$. The eigenvalues $\lambda_1^{\downarrow}(M), \lambda_2^{\downarrow}(M), \ldots, \lambda_n^{\downarrow}(M)$ of a self-adjoint matrix $M \in \mathbb{K}^{n\times n}$ are arranged in nonincreasing order, so that $\lambda_1^{\downarrow}(M) \geq \lambda_2^{\downarrow}(M) \geq \cdots \geq \lambda_n^{\downarrow}(M)$. The squared Frobenius norm $\|M\|_F^2 = \sum_{i,j=1}^n |M_{i,j}|^2$ of a matrix $M \in \mathbb{K}^{n\times n}$ can also be written as $\|M\|_F^2 = \sum_{k=1}^n \lambda_k^{\downarrow}(M)^2$. We use the letter B to represent a Seidel matrix, i.e., a self-adjoint matrix $B \in \mathbb{K}^{n\times n}$ with $B_{i,i}=0$ for all $i \in [\![1:n]\!]$ and $|B_{i,j}|=1$ for all $i \neq j \in [\![1:n]\!]$ — in the case $\mathbb{K}=\mathbb{R}$, these matrices are often called Seidel adjacency matrices. The set of $n\times n$ Seidel matrices is denoted by $\mathcal{S}_{\mathbb{K}}^{n\times n}$. We use the letter A to represent a matrix of the form $A = I_n + B$ where $B \in \mathcal{S}_{\mathbb{K}}^{n\times n}$, i.e., a self-adjoint matrix with the diagonal entries equal to one and off-diagonal entries having a modulus (or absolute value) equal to one.

2 Conversion of Maximal Relative Projection Constants

In this section, we highlight two alternative expressions for $\lambda_{\mathbb{K}}(m,N)$ that turn out to be useful for establishing some properties of the maximal relative projection constants, e.g. the properties listed in Proposition 2 below. These expressions are not new: the $\leq$-part of (3) uses trace duality and dates back to [13] (combining Proposition 2.2 with (3.7) on page 260); in the case $\mathbb{K} = \mathbb{R}$, both (3) and (4) are proved in [5] (combining Theorem 2.2 and Theorem 2.1) and again explicitly stated in [4] (combining Theorem 2.1 with Lemma 2.2). We propose different arguments that are also valid in the case $\mathbb{K} = \mathbb{C}$. Specifically, our argument for the $\geq$-part of (3) sidesteps Lagrange multipliers and Chalmers–Metcalf operators E_P. Besides, the identification of (3) and (4) is simplified.

Theorem 1. For integers $N \geq m$, one has

$$\lambda_{\mathbb{K}}(m,N) = \max\left\{ \sum_{i,j=1}^{N} t_i t_j |UU^*|_{i,j} : \ t \in \mathbb{R}_+^N, \|t\|_2 = 1, \ U \in \mathbb{K}^{N\times m}, U^*U = I_m \right\} \tag{3}$$

$$= \max\left\{ \sum_{k=1}^{m} \lambda_k^{\downarrow}(TAT) : \ T = \operatorname{diag}(t), t \in \mathbb{R}_+^N, \|t\|_2 = 1, \ A = I_N + B, B \in \mathcal{S}_{\mathbb{K}}^{N\times N} \right\}. \tag{4}$$

Proof. The justification of (3) is deferred to Appendix A to avoid digressing into unnecessary technicalities. We only highlight here a quick way to identify (3) and (4). For a fixed $t \in \mathbb{R}_+^N$ with $\|t\|_2 = 1$, writing $T := \operatorname{diag}(t)$, we have

$$\begin{aligned}
&\max\left\{ \sum_{i,j=1}^{N} t_i t_j |UU^*|_{i,j} : \ U \in \mathbb{K}^{N\times m}, U^*U = I_m \right\} \\
&= \max\left\{ \sum_{i,j=1}^{N} t_i t_j A_{i,j} \overline{(UU^*)_{i,j}} : \ U \in \mathbb{K}^{N\times m}, U^*U = I_m, \ A \in \mathbb{K}^{N\times N}, A^* = A, |A_{i,j}| = 1, A_{i,i} = 1 \right\} \\
&= \max\left\{ \operatorname{tr}((TAT)^* UU^*) : \ U \in \mathbb{K}^{N\times m}, U^*U = I_m, A = I_N + B, B \in \mathcal{S}_{\mathbb{K}}^{N\times N} \right\} \\
&= \max\left\{ \operatorname{tr}(U^* TATU) : \ U \in \mathbb{K}^{N\times m}, U^*U = I_m, A = I_N + B, B \in \mathcal{S}_{\mathbb{K}}^{N\times N} \right\} \\
&= \max\left\{ \sum_{k=1}^{m} \lambda_k^{\downarrow}(TAT) : \ A = I_N + B, B \in \mathcal{S}_{\mathbb{K}}^{N\times N} \right\},
\end{aligned}$$

where the last step is a known variational characterization of the sum of the m largest eigenvalues, see e.g. [2, Problem III.6.11]. Note that the maximum over U occurs when the m columns of U

are orthonormal eigenvectors of TAT associated with the eigenvalues $\lambda_1^\downarrow(TAT) \geq \cdots \geq \lambda_m^\downarrow(TAT)$. Taking the maximum over all t's concludes the proof. $\square$

A lower bound for $\lambda_{\mathbb{K}}(m,N)$ plays a central role throughout this article — it arises by making the particular choice $t = [1,\ldots,1]^\top/\sqrt{N}$ in (3)-(4). We denote it by $\mu_{\mathbb{K}}(m,N)$ and call it quasimaximal relative projection constant (even though it is not intrinsically a projection constant). Precisely,

$$\mu_{\mathbb{K}}(m,N) \leq \lambda_{\mathbb{K}}(m,N), \tag{5}$$

where $\mu_{\mathbb{K}}(m,N)$ satisfies

$$\mu_{\mathbb{K}}(m,N) = \frac{1}{N} \max\left\{ \sum_{i,j=1}^{N} |UU^*|_{i,j} : \; U \in \mathbb{K}^{N\times m}, U^*U = I_m \right\} \tag{6}$$

$$= \frac{1}{N} \max\left\{ \sum_{k=1}^{m} \lambda_k^\downarrow(A) : \; A = I_N + B, B \in \mathcal{S}_{\mathbb{K}}^{N\times N} \right\}. \tag{7}$$

Although the quantity $\mu_{\mathbb{R}}(m,N)$ is in general a strict lower bound for the quantity $\lambda_{\mathbb{R}}(m,N)$ (see Section 3), they do share some common properties listed below. These properties are proved using the expressions (3)-(4) and (6)-(7), even if several of them (e.g. 2)α) and 4)α)) could be seen directly from (1)-(2). Sometimes, the properties are better grasped in terms of the reduced quantities $\widetilde{\lambda}_{\mathbb{K}}(m,N) := \lambda_{\mathbb{K}}(m,N) - m/N$ and $\widetilde{\mu}_{\mathbb{K}}(m,N) := \mu_{\mathbb{K}}(m,N) - m/N$, i.e.,

$$\widetilde{\mu}_{\mathbb{K}}(m,N) = \frac{1}{N} \max\left\{ \sum_{\substack{i,j=1\\ i\neq j}}^{N} |UU^*|_{i,j} : \; U \in \mathbb{K}^{N\times m}, U^*U = I_m \right\} \tag{8}$$

$$= \frac{1}{N} \max\left\{ \sum_{k=1}^{m} \lambda_k^\downarrow(B) : \; B \in \mathcal{S}_{\mathbb{K}}^{N\times N} \right\}. \tag{9}$$

The latter expression readily implies that $\widetilde{\mu}_{\mathbb{K}}(1,N) = (N-1)/N$ (by Gershgorin theorem) and that $\widetilde{\mu}_{\mathbb{K}}(N,N) = 0$ (by the zero-trace of Seidel matrices), yielding the values $\mu_{\mathbb{K}}(1,N) = 1$ and $\mu_{\mathbb{K}}(N,N) = 1$. This matches the values $\lambda_{\mathbb{K}}(1,N) = 1$ and $\lambda_{\mathbb{K}}(N,N) = 1$. Note also that the inequality $\mu_{\mathbb{K}}(m,N) \geq 1$ holds in general, as can be seen by choosing A as the matrix with all 1's in (7).

Proposition 2. The maximal relative projection constants and quasimaximal relative projection constants have the following properties:

1) Real vs. complex:
   α) $\lambda_{\mathbb{R}}(m,N) \leq \lambda_{\mathbb{C}}(m,N) \leq 2\lambda_{\mathbb{R}}(m,2N)$, $\quad$ β) $\mu_{\mathbb{R}}(m,N) \leq \mu_{\mathbb{C}}(m,N) \leq 2\mu_{\mathbb{R}}(m,2N)$.

2) Symmetry in m:
   α) $\lambda_{\mathbb{K}}(m,N) - 1 \leq \lambda_{\mathbb{K}}(N-m,N) \leq \lambda_{\mathbb{K}}(m,N) + 1$, $\quad$ β) $\widetilde{\mu}_{\mathbb{K}}(N-m,N) = \widetilde{\mu}_{\mathbb{K}}(m,N)$.

3) Behavior with m: for $m \leq N-1$,
   α) $\lambda_{\mathbb{K}}(m+1,N) \leq \dfrac{m+1}{m}\,\lambda_{\mathbb{K}}(m,N)$, β) $\mu_{\mathbb{K}}(m+1,N) \leq \dfrac{m+1}{m}\,\mu_{\mathbb{K}}(m,N)$.

4) Behavior with N:
   α) $\lambda_{\mathbb{K}}(m,N) \leq \lambda_{\mathbb{K}}(m,N+1)$.

Remark 3. Before turning to the proof of Proposition 2, a few comments are in order:

- The inequalities in 1)α) are probably not sharp, but we emphasize that the leftmost inequality cannot be an equality and that the factor 2 in the rightmost inequality cannot be replaced by 1. Indeed, anticipating on Theorem 5, if $\lambda_{\mathbb{R}}(m,N)$ and $\lambda_{\mathbb{C}}(m,N)$ were equal, then the existence of equiangular tight frames consisting of N unit vectors in $\mathbb{K}^m$ would be independent of whether $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, which is not so. Moreover, $\lambda_{\mathbb{C}}(m,N)$ can exceed $\lambda_{\mathbb{R}}(m,2N)$, for instance when $m=2$ and $N=4$, since $\lambda_{\mathbb{C}}(2,4)=(1+\sqrt{3})/2 \approx 1.3660$ (anticipating again on Theorem 5) and $\lambda_{\mathbb{R}}(2,8)=4/3 \approx 1.3333$ (anticipating on Grünbaum conjecture). A similar observation is valid for the inequalities in 1)β).

- The strict symmetry of $\widetilde{\mu}(m,N)$ in 2)β) is not shared by $\widetilde{\lambda}(m,N)$. Indeed, the contrary would mean equality between $\lambda_{\mathbb{K}}(m,N)-m/N$ and $\lambda_{\mathbb{K}}(N-m,N)-1+m/N$. But such an equality does not hold, e.g. for $\mathbb{K}=\mathbb{R}$, $N=5$, and $m=2$, as shown by a direct calculation using the values $\lambda_{\mathbb{R}}(2,3)=4/3$ and $\lambda_{\mathbb{R}}(3,5)=(5+4\sqrt{2})/7$, see [5].

- The counterparts of 3)α-β) for $\widetilde{\lambda}_{\mathbb{K}}(m,N)$ and $\widetilde{\mu}_{\mathbb{K}}(m,N)$ are also valid. Precisely, it is simple to see that 3)α) is equivalent to $\widetilde{\lambda}_{\mathbb{K}}(m+1,N) \leq [(m+1)/m]\,\widetilde{\lambda}_{\mathbb{K}}(m,N)$ and that 3)α) is equivalent to $\widetilde{\mu}_{\mathbb{K}}(m+1,N) \leq [(m+1)/m]\,\widetilde{\mu}_{\mathbb{K}}(m,N)$.

Proof of Proposition 2. 1)α-β) follow from the expressions (3) and (6). The leftmost inequalities hold because any $U \in \mathbb{R}^{N\times m}$ also belongs to $\mathbb{C}^{N\times m}$. For the rightmost inequalities, we start by observing that any $U \in \mathbb{C}^{N\times m}$ with $U^*U = I_m$ satisfies, for all $i,j \in [\![1:N]\!]$,

$$\begin{aligned}
|UU^*|_{i,j} &= \sqrt{(\mathrm{Re}(U)\,\mathrm{Re}(U)^\top + \mathrm{Im}(U)\,\mathrm{Im}(U)^\top)_{i,j}^2 + (\mathrm{Re}(U)\,\mathrm{Im}(U)^\top - \mathrm{Im}(U)\,\mathrm{Re}(U)^\top)_{i,j}^2} \\
&\leq |\,\mathrm{Re}(U)\,\mathrm{Re}(U)^\top|_{i,j} + |\,\mathrm{Im}(U)\,\mathrm{Im}(U)^\top|_{i,j} + |\,\mathrm{Re}(U)\,\mathrm{Im}(U)^\top|_{i,j} + |\,\mathrm{Im}(U)\,\mathrm{Re}(U)^\top|_{i,j} \\
&= |VV^\top|_{i,j} + |VV^\top|_{i+N,j+N} + |VV^\top|_{i,j+N} + |VV^\top|_{i+N,j},
\end{aligned}$$

where the matrix $V \in \mathbb{R}^{2N\times m}$ is defined in block notation as $V := \begin{bmatrix} \mathrm{Re}(U) \\ \hline \mathrm{Im}(U) \end{bmatrix}$. It satisfies $V^\top V = I_m$. Summing over all $i,j \in [\![1:N]\!]$ and passing to the maxima gives $N\mu_{\mathbb{C}}(m,N) \leq 2N\mu_{\mathbb{R}}(m,2N)$, which yields the rightmost inequality in 1)β). As for the leftmost inequality in 1)α), given $t \in \mathbb{R}_+^N$ with $\|t\|_2 = 1$, we introduce $\tau \in \mathbb{R}_+^{2N}$ defined in block notation by $\tau = \dfrac{1}{\sqrt{2}}\begin{bmatrix} t \\ \hline t \end{bmatrix}$, so that $\|\tau\|_2 = 1$.

We then derive that

$$\sum_{i,j=1}^{N} t_i t_j |UU^*|_{i,j} \leq \sum_{i,j=1}^{N} \Big(t_i t_j |VV^\top|_{i,j} + t_i t_j |VV^\top|_{i+N,j+N} + t_i t_j |VV^\top|_{i,j+N} + t_i t_j |VV^\top|_{i+N,j} \Big)$$
$$= 2 \sum_{i,j=1}^{2N} \tau_i \tau_j |VV^\top|_{i,j} \leq 2\lambda_{\mathbb{R}}(m, 2N).$$

Passing to the maxima over U and t gives $\lambda_{\mathbb{C}}(m,N) \leq 2\lambda_{\mathbb{R}}(m,2N)$, as announced.

To see why 2)β) holds, we start by observing that, for any $B \in \mathcal{S}_{\mathbb{K}}^{N\times N}$,

$$\sum_{k=1}^{m} \lambda_k^{\downarrow}(B) = -\sum_{k=m+1}^{N} \lambda_k^{\downarrow}(B) = \sum_{k=1}^{N-m} \lambda_k^{\downarrow}(-B), \qquad \text{so} \qquad \frac{1}{N}\sum_{k=1}^{m} \lambda_k^{\downarrow}(B) \leq \widetilde{\mu}_{\mathbb{K}}(N-m, N),$$

because $-B \in \mathcal{S}_{\mathbb{K}}^{N\times N}$, too. Taking the maximum over B yields $\widetilde{\mu}_{\mathbb{K}}(m,N) \leq \widetilde{\mu}_{\mathbb{K}}(N-m,N)$. Exchanging the role of m and $N-m$ leads to the reversed inequality, and in turn to 2)β). Similarly, to prove 2)α), it is enough to establish that $\lambda_{\mathbb{K}}(m,N) \leq 1 + \lambda_{\mathbb{K}}(N-m,N)$. To this end, we write

$$\lambda_{\mathbb{K}}(m,N) = \sum_{i,j=1}^{N} t_i t_j |UU^*|_{i,j} \qquad \text{for } t \in \mathbb{R}_+^N \text{ with } \|t\|_2 = 1 \text{ and } U \in \mathbb{K}^{N\times m} \text{ with } U^*U = I_m.$$

The m columns of U form an orthonormal system of $\mathbb{K}^N$ that we complete to form an orthonormal basis of $\mathbb{K}^N$, thus introducing a matrix $V \in \mathbb{K}^{N\times(N-m)}$ such that $W := \Big[U \Big| V\Big]$ is a unitary matrix. Then $WW^* = I_N$ reads $UU^* + VV^* = I_N$. It follows that

$$\lambda_{\mathbb{K}}(m,N) = \sum_{i=1}^{N} t_i^2 (UU^*)_{i,i} + \sum_{i,j=1, i\neq j}^{N} t_i t_j |UU^*|_{i,j} = \sum_{i=1}^{N} t_i^2 (1 - VV^*)_{i,i} + \sum_{i,j=1, i\neq j}^{N} t_i t_j |VV^*|_{i,j}$$
$$= \sum_{i=1}^{N} t_i^2 + \sum_{i,j=1}^{N} t_i t_j |VV^*|_{i,j} - 2\sum_{i=1}^{N} t_i^2 (V^*V)_{i,i} \leq 1 + \lambda_{\mathbb{K}}(N-m,N),$$

which is the inequality we were targeting in order to establish 2)α).

Both 3)α) and 3)β) result from the observation $(\lambda_1^{\downarrow} + \cdots + \lambda_{m+1}^{\downarrow})/(m+1) \leq (\lambda_1^{\downarrow} + \cdots + \lambda_m^{\downarrow})/m$, valid for all real numbers $\lambda_1^{\downarrow} \geq \lambda_2^{\downarrow} \geq \cdots \geq \lambda_N^{\downarrow}$, when it is applied in (4) and (7).

4)α) follows from the expression (3) simply by appending a row of zeros to any $U \in \mathbb{K}^{N\times m}$ satisfying $U^*U = I_m$. $\square$

We conjecture that the list from Proposition 2 could be completed by two additional properties. The first conjectured property concerns the behavior with N of the quantity $\widetilde{\mu}_{\mathbb{K}}(m,N)$. Note that 4)$\alpha$), stating that $\lambda_{\mathbb{K}}(m,N)$ increases with N, implies that the reduced quantity $\widetilde{\lambda}_{\mathbb{K}}(m,N) = \lambda_{\mathbb{K}}(m,N) - m/N$ also increases with N. But a similar behavior can be expected only for $\widetilde{\mu}_{\mathbb{K}}(m,N)$, since the column $m = 2$ of Table 1 in Appendix B reveals that $\mu_{\mathbb{K}}(m,N)$ does not increase with N.

Conjecture 1. Behavior with N:

$$\widetilde{\mu}_{\mathbb{K}}(m,N) \leq \widetilde{\mu}_{\mathbb{K}}(m,N+1).$$

The second conjectured property concerns the behavior with m of the quantities $\widetilde{\lambda}_{\mathbb{K}}(m,N)$ and $\widetilde{\mu}_{\mathbb{K}}(m,N)$, which we expect to be maximized at $m = \lfloor N/2 \rfloor$ or $m = \lceil N/2 \rceil$. Note that a similar statement appears to be invalid for $\lambda_{\mathbb{K}}(m,N)$ and $\mu_{\mathbb{K}}(m,N)$. Tables 1 and 2 indicate that they should be maximized at $m = \lceil N/2 \rceil + 1$ instead when $N > 3$.

Conjecture 2. Behavior with m, bis: for $m \leq N/2 - 1$,

$$\alpha)\ \widetilde{\lambda}_{\mathbb{K}}(m,N) \leq \widetilde{\lambda}_{\mathbb{K}}(m+1,N), \qquad \beta)\ \widetilde{\mu}_{\mathbb{K}}(m,N) \leq \widetilde{\mu}_{\mathbb{K}}(m+1,N),$$

while the inequalities are reversed for $m \geq N/2$.

Given $m \leq N/2 - 1$, the result $\beta)$ would be clear if the Seidel matrix B_m attaining the maximum in (9) had its $(m+1)$st eigenvalue nonnegative. Unfortunately, in the case $\mathbb{K} = \mathbb{R}$ at least, the computations described in Section 3 suggest that $\lambda^{\downarrow}_{m+1}(B_m) < 0$ (as well as $\lambda^{\downarrow}_{m}(B_m) > 0$). We also point out that the computations hint that the inequality $\widetilde{\lambda}_{\mathbb{R}}(m,N) \leq \widetilde{\mu}_{\mathbb{R}}(m+1,N)$ could hold, which would immediately imply both $\alpha)$ and $\beta)$ in the case $\mathbb{K} = \mathbb{R}$.

3 Computation of Real Maximal Relative Projection Constants

In this section, which deals only with the real case, we present methods to calculate quasimaximal relative projection constants exactly and maximal relative projection constant approximately (at least). The methods, implemented in MATLAB, can be found in the reproducible accompanying this article. For small values of N, the results of our investigations are displayed in Tables 1 and 2 found in Appendix B. In the complex case, there is also a way to compute quasimaximal and maximal relative projection constants approximately. This is included in the reproducible file, but not incorporated in the main text because of theoretical uncertainty.

Computing quasimaximal relative projection constants: We simply determine $\mu_{\mathbb{R}}(m,N)$ as the value of the maximum in (7), which is possible because the set $\mathcal{S}_{\mathbb{R}}^{N \times N}$ of real Seidel matrices is finite (as a reminder, it consists of all symmetric real matrices with diagonal entries equal to zero and off-diagonal entries equal to -1 or $+1$). Hence, the quantity $\mu_{\mathbb{R}}(m,N)$ can be calculated exactly — at least in theory, but in practice it can only be calculated for small values of N because the number of real Seidel matrices quickly becomes prohibitive when N grows. This number can be reduced significantly, as explained later in this section, but it still limited our computations to $N = 10$. We have obtained, for instance,

$$\mu_{\mathbb{R}}(3,5) \approx 1.5123.$$

This shows that $\mu_{\mathbb{R}}(m,N)$ is in general a strict lower bound for $\lambda_{\mathbb{R}}(m,N)$, as it is known [5] that

$$\lambda_{\mathbb{R}}(3,5) = \frac{5+4\sqrt{2}}{7} \approx 1.5224.$$

To remain convinced of the strict inequality $\lambda_{\mathbb{R}}(3,5) > \mu_{\mathbb{R}}(3,5)$ without verifying all the details of [5], the reader can perform any of the following tests (all included in the MATLAB reproducible): enter the space provided in [5, Theorem 3.6] and calculate its projection constant using the software MinProj presented in [9]; make numerous random choices for t and U in (3) to derive a lower bound for $\lambda_{\mathbb{R}}(3,5)$; or compute (a lower bound for) $\lambda_{\mathbb{R}}(3,5)$ based on the method described next.

Computing maximal relative projection constants: Determining $\lambda_{\mathbb{R}}(m,N)$ directly as the value of the maximum in (3) appears difficult because of the simultaneous optimization over both variables t and U. We simply propose an iterative scheme that alternates between optimizing over one variable while keeping the other fixed. Precisely, starting with $t^{(0)} = [1,\ldots,1]^\top/\sqrt{N}$, we construct sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$, $(t^{(n)})_{n\geq 1}$, and $(U_{(n)})_{n\geq 1}$ as follows:

- α_n and $U_{(n)}$ are the maximum and maximizer in

$$\max\left\{ \sum_{i,j=1}^{N} t_i^{(n-1)} t_j^{(n-1)} |UU^*|_{i,j} : \;\; U \in \mathbb{R}^{N\times m}, U^*U = I_m \right\};$$

- β_n and $t^{(n)}$ are the maximum and maximizer in

$$\max\left\{ \sum_{i,j=1}^{N} t_i t_j |U_{(n)}U_{(n)}^*|_{i,j} : \; t \in \mathbb{R}_+^N, \|t\|_2 = 1 \right\}.$$

The advantage of this approach is that both steps are computable in the real case. Indeed, keeping in mind the identification of (3) and (4) in the proof of Theorem 1, we notice that determining α_n and $U_{(n)}$ involves computing the maximum over a finite set, since

$$\alpha_n = \max\left\{ \sum_{k=1}^{m} \lambda_k^{\downarrow}(T^{(n-1)}AT^{(n-1)}) : \; A = I_N + B, B \in \mathcal{S}_{\mathbb{K}}^{N\times N} \right\}$$

and $U_{(n)}$ is the matrix with columns equal to the orthonormal eigenvectors associated with the eigenvalues $\lambda_1^{\downarrow}(T^{(n-1)}AT^{(n-1)}) \geq \cdots \geq \lambda_m^{\downarrow}(T^{(n-1)}AT^{(n-1)})$ for the optimal matrix A. We then notice that determining β_n and $t^{(n)}$ requires another eigenvalue calculation, since

$$\beta_n = \lambda_1^{\downarrow}(|U_{(n)}U_{(n)}^*|)$$

and $t^{(n)}$ is the eigenvector associated with $\lambda_1^{\downarrow}(|U_{(n)}U_{(n)}^*|)$. In terms of theoretical guarantees, we can establish the following modest result, but we do expect that $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ genuinely converge to the true maximal relative projection constant rather than to a lower bound.

Theorem 4. The sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ are nondecreasing and converge to a common limit belonging to the interval $[\mu_{\mathbb{R}}(m,N), \lambda_{\mathbb{R}}(m,N)]$.

Proof. By the optimal properties of $t^{(n)}$ and $U_{(n-1)}$, we have

$$\sum_{i,j=1}^{N} t_i^{(n-1)} t_j^{(n-1)} |U_{(n)} U_{(n)}^*|_{i,j} \leq \sum_{i,j=1}^{N} t_i^{(n)} t_j^{(n)} |U_{(n)} U_{(n)}^*|_{i,j} \leq \sum_{i,j=1}^{N} t_i^{(n)} t_j^{(n)} |U_{(n+1)} U_{(n+1)}^*|_{i,j},$$

that is to say $\alpha_n \leq \beta_n \leq \alpha_{n+1}$. This shows that the sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ are both nondecreasing. Since they are bounded above by $\lambda_{\mathbb{R}}(m,N)$, they must be convergent. It is clear that they have the same limit, which is lower-bounded by $\alpha_1 = \mu_{\mathbb{R}}(m,N)$ and upper-bounded by $\lambda_{\mathbb{R}}(m,N)$. $\square$

Reducing the number of Seidel matrices: The computations of $\mu_{\mathbb{R}}(m,N)$ and $\lambda_{\mathbb{R}}(m,N)$ become quickly prohibitive when N increases. Indeed, we take the maximum over the set $\mathcal{S}_{\mathbb{R}}^{N\times N}$, which has huge cardinality $2^{N(N-1)/2}$. However, since we are only interested in the eigenvalues of matrices $T(I+B)T$, $B \in \mathcal{S}_{\mathbb{R}}^{N\times N}$ and $T \in \mathbb{R}^{N\times N}$ diagonal with $\|T\|_F = 1$, we shall avoid repeating matrices that have the same spectrum. There are two obvious operations on $\mathcal{S}_{\mathbb{K}}^{N\times N}$ that preserve the spectrum of $T(I+B)T$ up to a possible change of T:

- the multiplication on the left by a diagonal matrix D of -1's and $+1$'s and on the right by its inverse $D^{-1} = D$, since

$$T(I+DBD)T = (TD)(I+B)(DT) = D[T(I+B)T]D^{-1};$$

- the multiplication on the left by a permutation matrix P and on the right by its inverse P^{-1}, since

$$T(I+PBP^{-1})T = (TP)(I+B)(P^{-1}T) = P[(P^{-1}TP)(I+B)(P^{-1}TP)]P^{-1},$$

  where $P^{-1}TP$ is a diagonal matrix with $\|P^{-1}TP\|_F = 1$ (since its diagonal entries are those of T in a different order).

If one thinks of Seidel matrices as $(-1,0,1)$-adjacency matrices[1] of simple graphs, then the first operation corresponds to switching the connectivity of some vertices and the second operation corresponds to reordering the vertices. Now, given an $N\times N$ real Seidel matrix B, multiplying on the left and on the right by $D := \mathrm{diag}[1, B_{1,2}, \ldots, B_{1,N}]$, we can assume that B takes the form

$$B = \left[\begin{array}{c|ccc} 0 & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & B' & \\ 1 & & & \end{array}\right], \qquad \text{where } B' \text{ is an } (N-1)\times(N-1) \text{ real Seidel matrix.}$$

[1]The (i,j)th entry of a $(-1,0,1)$-adjacency matrix is 0 if $i=j$, 1 if $i \neq j$ and there is an edge connecting i and j, and -1 if $i \neq j$ and there is no edge connecting i and j.

The graph corresponding to this matrix B' can be chosen up to isomorphism (i.e., reordering of the vertices), since such an operation does not affect the first row and column of B. Thus, the number of matrices we need to consider has reduced from $2^{N(N-1)/2}$ to the number of nonisomorphic simple graphs on $N-1$ vertices, which is tabulated. Although this number still becomes prohibitive when N increases, for $N = 10$, at least, there is a substantial reduction from $35,184,372,088,832$ down to $274,668$, which makes this computation amenable. Moreover, the construction of all nonisomorphic simple graphs can be performed rather efficiently using a software called nauty made publicly available by the authors of [14] on their webpages. All in all, the previous considerations allowed us to compute the values of $\mu_{\mathbb{R}}(m, N)$ and of (lower bounds for) $\lambda_{\mathbb{R}}(m, N)$ for all $1 \le m \le N \le 10$. They are reported in Tables 1 and 2 displayed in Appendix B.

4 Connection with Equiangular Tight Frames

Sets of equiangular lines have appeared earlier in the study of absolute projection constants (see e.g. [13, Theorem 1.1], but beware that [5] exposed a flaw in the argument which essentially amounts to identifying $\lambda_{\mathbb{K}}(m, N)$ and $\mu_{\mathbb{K}}(m, N)$). In the context of relative projection constants, it is the notion of equiangular tight frames that becomes pertinent. The article [12] contained the important ingredients, even though it took a different perspective. The purpose of this section is to formulate explicitly the connection between (quasi)maximal relative projection constants and equiangular tight frames and to derive consequences about the former from facts about the latter. First of all, we recall that a system of unit (i.e., ℓ_2-normalized) vectors $f_1, \ldots, f_N$ in $\mathbb{K}^m$ is called equiangular if

$$|\langle f_i, f_j \rangle| = c \qquad \text{for all } i \neq j \in [\![1:N]\!].$$

It is called a tight frame if the matrix $F := \begin{bmatrix} f_1 & | & \cdots & | & f_N \end{bmatrix} \in \mathbb{K}^{m \times N}$ satisfies

$$FF^* = \frac{N}{m} I_m.$$

A usual definition would free the constant in front of I_m, but here the normalization of the f_i's forces it to be N/m, as readily seen by looking at $\mathrm{tr}(FF^*) = \mathrm{tr}(F^*F)$. Evidently, the system $(f_1, \ldots, f_N)$ is called an equiangular tight frame if it is both equiangular and a tight frame. Equiangular tight frames are exactly the systems of unit vectors for which the Welch bound is met, i.e., for which the inequality

$$\max_{i \neq j \in [\![1:N]\!]} |\langle f_i, f_j \rangle| \ge \sqrt{\frac{N-m}{m(N-1)}}$$

becomes an equality (see [10, Theorem 5.7] among other possible references). Thus, the existence of an equiangular tight frame consisting of N unit vectors is equivalent to the existence of $U \in \mathbb{K}^{N \times m}$

($U = \sqrt{m/N}\, F^*$) such that

$$(UU^*)_{i,i} = \frac{m}{N},\ i \in [\![1:N]\!], \qquad |UU^*|_{i,j} = \frac{m}{N}\sqrt{\frac{N-m}{m(N-1)}},\ i \neq j \in [\![1:N]\!], \qquad U^*U = I_m, \tag{10}$$

where the last condition is actually superfluous because it is a consequence of the first two. We are now prepared to state the main result of this section.

Theorem 5. Given integers $N \geq m$, the following properties are equivalent:

(i) there is an equiangular tight frame consisting of N unit vectors in $\mathbb{K}^m$,

(ii) $\mu_{\mathbb{K}}(m,N) = \dfrac{m}{N}\left(1 + \sqrt{\dfrac{(N-1)(N-m)}{m}}\right)$,

(iii) $\lambda_{\mathbb{K}}(m,N) = \dfrac{m}{N}\left(1 + \sqrt{\dfrac{(N-1)(N-m)}{m}}\right)$.

Remark 6. When Condition (i) is met with $\mathbb{K} = \mathbb{R}$ and $N \neq 2m$, the quantity $(N-1)(N-m)/m$ is the square of an odd integer (see [20, Theorem A]), so that the maximal relative projection constant $\lambda_{\mathbb{R}}(m,N)$ is a rational number in this case.

Remark 7. It is possible for $\mu_{\mathbb{K}}(m,N)$ and $\lambda_{\mathbb{K}}(m,N)$ to be equal, yet there are no equiangular tight frames consisting of N unit vectors in $\mathbb{K}^m$. For instance, the value $\mu_{\mathbb{R}}(2,9) = 4/3$ is provided by our computations, while the value $\lambda_{\mathbb{R}}(2,9) = 4/3$ results from a proof of Grünbaum conjecture (see Remark 11). However, equiangular tight frames in $\mathbb{R}^2$ cannot consist of 9 unit vectors, as the maximal number of unit vectors in this case is 3. We also point out that if equality between $\mu_{\mathbb{K}}(m,N)$ and $\lambda_{\mathbb{K}}(m,N)$ occurs, then there is an m-dimensional subspace of ℓ_∞^N maximizing the projection constant for which the orthogonal projection is a minimal projection. This statement is proved in Appendix C.

Proof of Theorem 5. Without restriction on m and N, invoking (5) and [12, Theorem 1] yields

$$\mu_{\mathbb{K}}(m,N) \leq \lambda_{\mathbb{K}}(m,N) \leq \frac{m}{N}\left(1 + \sqrt{\frac{(N-1)(N-m)}{m}}\right). \tag{11}$$

We also refer to Appendix D for a new proof based on the expression (4).

(i) $\Rightarrow$ (ii): Picking a matrix $U \in \mathbb{K}^{N\times m}$ satisfying (10), the expression (6) for $\mu_{\mathbb{K}}(m,N)$ gives

$$\mu_{\mathbb{K}}(m,N) \geq \frac{1}{N}\left(N\frac{m}{N} + (N^2-N)\frac{m}{N}\sqrt{\frac{N-m}{m(N-1)}}\right) = \frac{m}{N}\left(1 + \sqrt{\frac{(N-1)(N-m)}{m}}\right). \tag{12}$$

Putting the estimates (11) and (12) together establishes Condition (ii).
(ii) $\Rightarrow$ (iii): This is a direct consequence of (11).

(iii) $\Rightarrow$ (i): According to [12, Theorem 2], Condition (iii) implies the existence of a self-adjoint[2] matrix $C \in \mathbb{K}^{N \times N}$ such that $C^2 = I_N$, $C_{i,i} = 2m/N - 1$, and $|C_{i,j}| = 2/N\sqrt{m(N-m)/(N-1)}$. Note that $D := (C + I_N)/2$ represents the matrix of an orthogonal projection (since $D^* = D$ and $D^2 = D$) with rank equal to $\mathrm{tr}(D) = m$. This implies (look e.g. at the eigendecomposition of D) that $D = UU^*$ for some $U \in \mathbb{K}^{N \times m}$. The first two conditions of (10), namely $(UU^*)_{i,i} = m/N$ and $|UU^*|_{i,j} = m/N\sqrt{(N-m)/(m(N-1))}$ are automatically fulfilled, and the third condition, namely $U^*U = I_m$, occurs as a consequence. As mentioned above, these three conditions are equivalent to Condition (i) being fulfilled. $\square$

The question of existence of equiangular tight frames has been extensively studied (but so far not completely settled), so Theorem 5 provides the exact values of the maximal relative projection constants in a number of known situations (see [20, 3, 21] for table listing cases of existence). As a first example, since there always exists an equiangular tight frame consisting of $m+1$ unit vectors in $\mathbb{R}^m$ — the vertices of the m-simplex centered at the origin — one retrieves the value $\lambda_{\mathbb{R}}(m, m+1) = 2 - 2/(m+1)$ found e.g. in [5, Lemma 2.6]. As a second example, the small equiangular tight frames collected in [10, Exercises 5.5 and 5.6] yield

$$\lambda_{\mathbb{R}}(3,6) = \frac{1+\sqrt{5}}{2} \approx 1.6180, \qquad \lambda_{\mathbb{R}}(7,28) = \frac{5}{2} = 2.5.$$

$$\lambda_{\mathbb{C}}(2,4) = \frac{1+\sqrt{3}}{2} \approx 1.3660, \qquad \lambda_{\mathbb{C}}(3,9) = \frac{5}{3} \approx 1.6667.$$

More unexpectedly, Condition (ii) from Theorem 5 provides an original test for the existence of equiangular tight frames consisting of N unit vectors in $\mathbb{R}^m$, since Section 3 showed that the quantity $\mu_{\mathbb{R}}(m, N)$ is 'computable'. For instance, there is no equiangular tight frames consisting of 5 unit vectors in $\mathbb{R}^3$, since $\mu_{\mathbb{R}}(3,5) \approx 1.5123$ is strictly smaller than the value $(3+2\sqrt{6})/5 \approx 1.5798$ of the upper bound. All of this being said, the usefulness of Theorem 5 is counterbalanced by the fact that equiangular tight frames are rare. In particular, they cannot possess an arbitrarily large number of vectors. On this matter, there is some difference between the complex case and the real case, as further discussed below.

Complex case: An equiangular tight frame consisting of N unit vectors in $\mathbb{C}^m$ must satisfy $N \le m^2$. Zauner conjecture suggests that this maximal number can be attained for every positive integer m — at least, the numerical investigations of [18] indicate that it is so up to $m = 67$. Regardless, if maximal equiangular tight frames in $\mathbb{C}^m$ do exist for a specific m, then the expression of the maximal relative projection constant simplifies to

$$\lambda_{\mathbb{C}}\left(m, m^2\right) = \frac{1}{m} + \frac{(m-1)\sqrt{m+1}}{m}.$$

This would give a sequence $(\mathcal{V}_m)_{m \ge 1}$ of complex m-dimensional spaces such that

$$\lim_{m\to\infty} \frac{\lambda_{\mathbb{C}}(\mathcal{V}_m)}{\sqrt{m}} = 1.$$

[2] Self-adjointness was left out in the statement of the theorem, but it is implied by the conditions satisfied by C.

Such a result can genuinely be obtained if one relaxes the equiangularity constraint. This point was already observed in [11], and the following example provides a marginal improvement which increases the lower bound on $\lambda_{\mathbb{C}}(m,m^2)$ by the anecdotal quantity $(1-1/m)^2/\sqrt{m}$. Precisely, when $m \ge 5$ is a prime number, the proof of [10, Proposition 5.13] (originally from [19]) offers an explicit matrix $U \in \mathbb{C}^{m^2\times m}$ with $U^*U = I_m$ and

$$(UU^*)_{i,i} = \frac{1}{m},\ i \in [\![1:m^2]\!], \quad |UU^*|_{i,j} = \begin{cases} 0 & m(m-1) \text{ times}, \\ \dfrac{1}{m\sqrt{m}} & m(m-1)(m^2+m-1) \text{ times}, \end{cases} \ i \neq j \in [\![1:m^2]\!].$$

Therefore, the expression (6) for $\mu_{\mathbb{C}}(m,m^2)$ yields

$$\begin{aligned}\lambda_{\mathbb{C}}(m,m^2) \ge \mu_{\mathbb{C}}(m,m^2) &\ge \frac{1}{m^2}\left(m^2\frac{1}{m} + m(m-1)(m^2+m-1)\frac{1}{m\sqrt{m}}\right) \\ &= \frac{1}{m} + \frac{(m-1)(m^2+m-1)}{m^3}\sqrt{m}.\end{aligned}$$

Real case: An equiangular tight frame consisting of N unit vectors in $\mathbb{R}^m$ must satisfy the stronger estimate $N \le m(m+1)/2$. For $m \ge 3$, if this maximal number is attained, then $m+2$ must be the square of an odd integer. Among the possible values, $m=7$ and $m=23$ do showcase maximal equiangular tight frames, but $m=47$ does not (and neither do several other plausible m's, see [1]), while the situation remains uncertain for $m=79$, etc. Regardless, if maximal equiangular tight frames in $\mathbb{R}^m$ do exist for infinitely many m, then the expression of the maximal relative projection constant simplifies to

$$\lambda_{\mathbb{R}}\left(m, \frac{m(m+1)}{2}\right) = \frac{2}{m+1} + \frac{(m-1)\sqrt{m+2}}{m+1}.$$

This would give a increasing sequence $(\mathcal{V}_{m_q})_{q\ge1}$ of real m_q-dimensional spaces such that

$$\lim_{q\to\infty} \frac{\lambda_{\mathbb{R}}(\mathcal{V}_{m_q})}{\sqrt{m_q}} = 1. \tag{13}$$

Such a result can be obtained conditionally on an affirmative answer to Hadamard conjecture (which states that Hadamard matrices, i.e., matrices populated by ±1 and having orthogonal rows, do exist whenever their size is a multiple of 4). Indeed, the existence of equiangular tight frames consisting of $N_q := q^n(1+(q^n-1)/(q-1))$ unit vectors in $\mathbb{R}^{m_q}$, $m_q := q^{n-1}(q^n-1)/(q-1)$, was established in [8, Subsection 3.1.5] provided that the prime power q and the integer $n>1$ are such that $1+(q^n-1)/(q-1) = q^{n-1}+q^{n-2}+\cdots+q+2$ is the size of a Hadamard matrix. So, accepting Hadamard conjecture, one can simply choose $n=3$ and q to be one of the infinitely many prime numbers congruent to 1 modulo 4, because $q^2+q+2 \equiv 1^2+1+2 \equiv 0 \pmod 4$. Then, for the sequences $(m_q)_{q\ge1}$ and $(N_q)_{q\ge1}$, Theorem 5 yields

$$\frac{\lambda_{\mathbb{R}}(m_q,N_q)}{\sqrt{m_q}} = \frac{\sqrt{m_q}}{N_q}\left(1+\sqrt{\frac{(N_q-1)(N_q-m_q)}{m_q}}\right) \underset{q\to\infty}{\sim} \frac{\sqrt{m_q}}{N_q}\sqrt{\frac{N_q^2}{m_q}} = 1.$$

Note that Hadamard matrices are known to exist when their size is a power of 2, but finding infinitely many admissible q seems unlikely in this situation because the intended condition reduces to $(q^n-1)/(q-1) = 2^k - 1$, which is believed to have only finitely many solutions according to Goormaghtigh conjecture. However, if one again relaxes the equiangularity constraint, then we can genuinely find a sequence $(\mathcal{V}_{m_q})_{q\geq 1}$ of real m_q-dimensional spaces for which (13) holds. To do so, we borrow arguments from Section 3 of [15][3], which relies on Taylor graphs to establish that, as soon as q is an odd prime power, there exists a graph of order $N_q := q^3$ whose adjacency matrix[4] $G \in \{0,1\}^{N_q \times N_q}$ has eigenvalues $(q+1)(q^2-1)/2$ with multiplicity 1, $(q^2-1)/2$ with multiplicity $q(q-1)$, and $-(q+1)/2$ with multiplicity $(q-1)(q^2+1)$. Note that, if J denotes the matrix whose entries all equal 1, then $B := 2G + I - J$ is an $N_q \times N_q$ Seidel matrix. Setting $m_q := q^2 - q + 1$, we have, for $A := I + B$,

$$\begin{aligned}\sum_{k=1}^{m_q} \lambda_k^{\downarrow}(A) = \sum_{k=1}^{m_q} \lambda_k^{\downarrow}(2(G+I)-J) \geq \sum_{k=1}^{m_q} \lambda_k^{\downarrow}(2(G+I)) - \sum_{k=1}^{m_q} \lambda_k^{\downarrow}(J) = 2\sum_{k=1}^{m_q} \lambda_k^{\downarrow}(G) + 2m_q - N_q \\ = (q+1)(q^2-1) + (q^2-1)q(q-1) + 2m_q - N_q = (q^2-1)(q^2+1) + 2m_q - N_q.\end{aligned}$$

We finally derive (13) by observing that

$$\frac{\lambda_{\mathbb{R}}(m_q,N_q)}{\sqrt{m_q}} \geq \frac{\mu_{\mathbb{R}}(m_q,N_q)}{\sqrt{m_q}} \geq \frac{(q^2-1)(q^2+1)/N_q + 2m_q/N_q - 1}{\sqrt{m_q}} \underset{q\to\infty}{\sim} \frac{q}{\sqrt{q^2}} = 1.$$

5 Refined Bounds on Maximal Relative Projection Constants

In Section 4, we have given evidence that, for some positive integers m, the Kadec–Snobar upper bound on the maximal relative projection constant can be reversed as $\lambda_{\mathbb{K}}(m,N) \geq c\sqrt{m}$ with N depending more than linearly on m and with a constant c arbitrarily close to 1. This section shows, roughly speaking, that the lower bound $\lambda_{\mathbb{K}}(m,N) \geq c\sqrt{m}$ is in fact valid for all positive integers m with N depending only linearly on m. The constant c cannot exceed $\sqrt{(\kappa-1)/\kappa} < 1$ in case $N \leq \kappa m$, though. Indeed, as shown in [12], the quantity appearing in Theorem 5 is always an upper bound for $\lambda_{\mathbb{K}}(m,N)$, so that

$$\lambda_{\mathbb{K}}(m,N) \leq \lambda_{\mathbb{K}}(m,\lfloor \kappa m \rfloor) \leq \frac{m}{\lfloor \kappa m \rfloor}\left(1 + \sqrt{\frac{(\lfloor \kappa m \rfloor - 1)(\lfloor \kappa m \rfloor - m)}{m}}\right) \underset{m\to\infty}{\sim} = \sqrt{\frac{\kappa-1}{\kappa}}\sqrt{m}.$$

[3] The article [15] studies the quantity $\tau(m,N) = \max\{\sum_{k=1}^m \lambda_k^{\downarrow}(G) :\ G$ adjacency matrix of graph of order $n\}/N$. It is strongly related to our quantity $\mu(m,N)$ — one can show that $|\mu(m,N) - 2\tau(m,N) - 2m/N| \leq 1$. It is basically [15, Theorem 1.4] that yields (13). The upper bound $\tau(m,N) \leq (\sqrt{m}+1)/2$ found in [15, Theorem 1.3] follows quickly from the Kadec–Snobar estimate $\mu_{\mathbb{K}}(m,N) \leq \lambda_{\mathbb{R}}(m,N) \leq \sqrt{m}$, and our stronger bound (11) in fact improves on [15, Theorem 1.3] in case N does not tend to infinity.

[4] As opposed to the $(-1,0,1)$-adjacency matrix encountered before, for this more classical notion of adjacency matrix, the (i,j)th entry is 0 if there is no edge connecting i and j and 1 if there is an edge connecting i and j.

The argument we propose consists in bounding $\widetilde{\mu}_{\mathbb{R}}(m,N)$ from below by making random, rather than optimal, choices for the matrices B in (9). The main ingredient is the semicircle law for the limiting distribution of eigenvalues of random symmetric matrices $B_N \in \mathbb{R}^{N\times N}$ whose diagonal entries equal 0 and whose off-diagonal entries equal -1 or $+1$, each with probability 1/2. Given the density function f defined by $f(x)=0$ for $|x|>2$ and by

$$f(x) = \frac{1}{2\pi}\sqrt{4-x^2} \qquad \text{for } |x| \le 2,$$

and with F denoting the complementary distribution function defined by

$$F(x) = \int_x^2 f(t)dt = \frac{\arccos(x/2)}{\pi} - \frac{x\sqrt{4-x^2}}{2} \qquad \text{for } |x|\le 2,$$

we invoke one of the earlier results (see [17]) which states that, for any $\varepsilon>0$ and any $y\in\mathbb{R}$,

$$h_{\varepsilon,y}(N) := \mathbb{P}\left(\left| \frac{\#\{\text{eigenvalues of } B_N/\sqrt{N} \text{ in } [y,\infty)\}}{N} - F(y) \right| > \varepsilon \right) \underset{N\to\infty}{\longrightarrow} 0. \tag{14}$$

Although more sophisticated results have appeared since then, this simple version is enough for us to establish the following intermediate fact.

Theorem 8. Given $\rho\in(0,1/2)$ and $\varepsilon\in(0,\min\{\rho/2,1/2-\rho\})$,

$$\frac{1}{N}\sum_{1\le k\le \rho N} \lambda_k^{\downarrow}(B_N) \ge F^{-1}(\rho-\varepsilon)(\rho-2\varepsilon)\sqrt{N}$$

for a proportion of real $N\times N$ Seidel matrices approaching 1 as $N\to\infty$.

In terms of (quasi)maximal relative projection constants, the result below follows immediately.

Corollary 9. Given $\rho\in(0,1/2)$ and $c\in(0,F^{-1}(\rho)\sqrt{\rho})$, if $m\ge m^*$ for some $m^*=m^*_{c,\rho}$, then

$$\lambda_{\mathbb{R}}(m,N) \ge \mu_{\mathbb{R}}(m,N) \ge c\sqrt{m}$$

as soon as $N\ge \rho^{-1}m$.

Proof of Theorem 8. Let $a=a_{\rho,\varepsilon}>0$ be chosen such that $F(a)=\rho-\varepsilon$. Let r_N denote the maximum index $i\in[\![1:N]\!]$ such that $\lambda_i^{\downarrow}(B_N/\sqrt{N})\ge 0$, and let s_N denote the maximum index $i\in[\![1:N]\!]$ such that $\lambda_i^{\downarrow}(B_N/\sqrt{N})\ge a$. Applying (14) to $y=0$ and $y=a$, we obtain that

- with failure probability $h_{\varepsilon,0}(N)$,
  $\left|\frac{r_N}{N}-\frac{1}{2}\right|<\varepsilon,$ i.e., $\left(\frac{1}{2}-\varepsilon\right)N\le r_N\le\left(\frac{1}{2}+\varepsilon\right)N,$ and in particular $r_N\ge\rho N$;
- with failure probability $h_{\varepsilon,a_{\rho,\varepsilon}}(N)$,
  $\left|\frac{s_N}{N}-F(a)\right|<\varepsilon,$ i.e., $(F(a)-\varepsilon)N\le s_N\le(F(a)+\varepsilon)N.$

So, with failure probability at most $h_{\varepsilon,0}(N)+h_{\varepsilon,a_{\rho,\varepsilon}}(N)$, we have $\lambda_k^{\downarrow}(B_N/\sqrt{N}) \geq \lambda_{r_N}^{\downarrow}(B_N/\sqrt{N}) \geq 0$ whenever $1 \leq k \leq \rho N$, as well as $(\rho - 2\varepsilon)N \leq s_N \leq \rho N$. It follows that

$$\sum_{1 \leq k \leq \rho N} \lambda_k^{\downarrow}(B_N/\sqrt{N}) \geq \sum_{k=1}^{s_N} \lambda_k^{\downarrow}(B_N/\sqrt{N}) \geq \sum_{k=1}^{s_N} a \geq a(\rho - 2\varepsilon)N.$$

We arrive at the desired conclusion after dividing by $\sqrt{N}$. □

6 Concluding Remarks

To close this article, we show how the reformulations (4) and (7) of maximal and quasimaximal relative projection constants in terms of eigenvalues can be elegantly exploited to retrieve results on minimal projections or instead to turn results on minimal projections into results in matrix theory or graph theory.

Remark 10. As announced in the introduction, the Kadec–Snobar estimate $\lambda_{\mathbb{K}}(m, N) \leq \sqrt{m}$ can be derived from (4) in a simple way. For this purpose, one notices that, for $T = \mathrm{diag}(\mathrm{t})$ with $t \in \mathbb{R}_+^N$ and $\|t\|_2 = 1$ and for $A \in \mathbb{K}^{N \times N}$ with modulus-one entries, one has $\|TAT\|_F^2 = \sum_{i,j=1}^N |t_i A_{i,j} t_j|^2 = \sum_{i,j=1}^N t_i^2 t_j^2 = 1$, and consequently

$$\sum_{k=1}^m \lambda_k(TAT) \leq m^{1/2} \left[\sum_{k=1}^m \lambda_k(TAT)^2 \right]^{1/2} \leq m^{1/2} \left[\sum_{k=1}^N \lambda_k(TAT)^2 \right]^{1/2} = m^{1/2} \|TAT\|_F = m^{1/2}.$$

It now suffices to take the maximum over t and A. The stronger bound (11) on $\lambda_{\mathbb{K}}(m, N)$ can also be shown using (4), as detailed in Appendix D.

Remark 11. Grünbaum conjecture states that $\lambda_{\mathbb{R}}(2) = 4/3$ — which does not hold in the complex case, for $\lambda_{\mathbb{C}}(2, 4) = (1 + \sqrt{3})/2 > 4/3$. The proof proposed in [13] relied on an erroneous lemma, as pointed out in [5], but it was completed in [6]. Based on the different expressions of $\lambda_{\mathbb{R}}(m, N)$ given in this paper, namely (3) and (4), the result can be rephrased as either one of the statements that have nothing to do with projection constants:

- for every $U \in \mathbb{R}^{N \times 2}$ with $U^*U = I_2$,

$$\lambda_1^{\downarrow}(|UU^*|) \leq 4/3;$$

- for every graph of order N with Seidel adjacency matrix $B \in \{-1, 0, 1\}^{N \times N}$ and for every diagonal matrix $T \in \mathbb{R}^{N \times N}$ with unit Frobenius norm,

$$\lambda_1^{\downarrow}(T(I + B)T) + \lambda_2^{\downarrow}(T(I + B)T) \leq 4/3.$$

Remark 12. It is known [16, Theorem 3] that the matrices of the form UU^* with $U \in \mathbb{R}^{N\times m}$ and $U^*U = I_m$ are extreme points of the set $\{Z \in \mathbb{R}^{N\times N} : 0 \preceq Z \preceq I_N, \mathrm{tr}(Z) = m\}$. Hence the real maximal and quasimaximal relative projection constants also take the forms

$$\lambda_{\mathbb{R}}(m,N) = \max\left\{ \sum_{i,j=1}^{N} t_i t_j |Z|_{i,j} :\ t \in \mathbb{R}_+^N, \|t\|_2 = 1, Z \in \mathbb{R}^{N\times N} : 0 \preceq Z \preceq I_N, \mathrm{tr}(Z) = m \right\},$$

$$\mu_{\mathbb{R}}(m,N) = \frac{1}{N}\max\left\{ \sum_{i,j=1}^{N} |Z|_{i,j} :\ Z \in \mathbb{R}^{N\times N} : 0 \preceq Z \preceq I_N, \mathrm{tr}(Z) = m \right\}.$$

Grünbaum conjecture/theorem can then be rephrased as

- for every $Z \in \mathbb{R}^{N\times N}$ with $0 \preceq Z \preceq I_N$ and $\mathrm{tr}(Z) = 2$,

$$\lambda_1^{\downarrow}(|Z|) \le 4/3.$$

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Appendix A: Proof of the Expression (3) for the Maximal Relative Projection Constant

We justify here the expression for the maximal relative projection constant given in (3), namely

$$\lambda_{\mathbb{K}}(m,N) = \max\left\{ \sum_{i,j=1}^{N} t_i t_j |UU^*|_{i,j} :\ t \in \mathbb{R}_+^N, \|t\|_2 = 1,\ U \in \mathbb{K}^{N\times m}, U^*U = I_m \right\}.$$

We start by proving the $\leq$-part — as mentioned earlier, this is a rephrasing of existing arguments. Let $\mathcal{V}_m$ be an m-dimensional subspace of ℓ_∞^N and let $V \in \mathbb{K}^{N\times m}$ be a matrix whose columns form an orthonormal basis of $\mathcal{V}_m$, so that $V^*V = I_m$. We draw attention to the easy observations that (matrices of) operators form ℓ_∞^N into $\mathcal{V}_m$ take the form VW^* for some $W \in \mathbb{K}^{N\times m}$ and that (matrices of) projections from ℓ_∞^N onto $\mathcal{V}_m$ take the form VW^* for some $W \in \mathbb{K}^{N\times m}$ satisfying $W^*V = I_m$. We shall first prove the expression (experimentally validated in the MATLAB reproducible)

$$\lambda(\mathcal{V}_m, \ell_\infty^N) = \max\left\{ \mathrm{Re}\,\mathrm{tr}(C) :\ M \in \mathbb{K}^{N\times N}, \|M\|^*_{\infty\to\infty} = 1, C \in \mathbb{K}^{m\times m}, M^*V = VC \right\}, \tag{15}$$

where the norm on $\mathbb{K}^{N\times N}$ that is dual to the operator norm on ℓ_∞^N is given by

$$\|M\|^*_{\infty\to\infty} = \sum_{i=1}^{N} \max_{j\in[\![1:N]\!]} |M|_{i,j}.$$

To this end, we rely on the following equivalence: given a subspace $\mathcal{Y}$ of a normed space $\mathcal{X}$ and given $x \in \mathcal{X}\setminus\mathcal{Y}$, a vector $y \in \mathcal{Y}$ is a best approximation to x from $\mathcal{Y}$ if and only if there exists a linear functional $\eta \in \mathcal{X}^*$ vanishing on $\mathcal{Y}$ which satisfies $\eta(x) = \|x-y\|$ and $\|\eta\|^* = 1$. This equivalence is for instance crucial in the duality between Kolmogorov and Gelfand widths, as it yields (see e.g. [7, Theorem 1.3] or [10, p. 323-324] for details)

$$\min\left\{\|x-y\| :\ y \in \mathcal{Y}\right\} = \max\left\{\eta(x) :\ \eta \in \mathcal{X}^*, \eta_{|\mathcal{Y}} = 0, \|\eta\|^* = 1\right\}.$$

In the present situation, with $\mathcal{X} := \mathbb{K}^{N\times N}$ and $P \in \mathcal{X}$ denoting a projection from ℓ_∞^N onto $\mathcal{V}_m$, the relative projection constant $\lambda(\mathcal{V}_m, \ell_\infty^N)$ equals $\min\{\|P-Q\|_{\infty\to\infty} :\ Q \in \mathcal{Y}\}$, where the space $\mathcal{Y} := \{VW^* :\ W \in \mathbb{K}^{N\times m} \text{ with } W^*V = 0\}$ of operators from ℓ_∞^N into $\mathcal{V}_m$ vanishing on $\mathcal{V}_m$ has dimension $Nm - m^2$. Using trace duality to represent the linear functionals on $\mathcal{X}$, we arrive at $\lambda(\mathcal{V}_m, \ell_\infty^N) = \max\{\mathrm{Re}\,\mathrm{tr}(M^*P), M \in \mathcal{Z}, \|M\|^*_{\infty\to\infty} = 1\}$, where $\mathcal{Z}$ is the space of dimension $N^2 - Nm + m^2$ defined by $\mathcal{Z} := \{M \in \mathbb{K}^{N\times N} : \mathrm{tr}(M^*Q) = 0 \text{ whenever } Q \in \mathcal{Y}\}$. We claim that $\mathcal{Z}$ can also be described as the space $\mathcal{Z}' := \{M \in \mathbb{K}^{N\times N} : M^*V = VC \text{ for some } C \in \mathbb{K}^{m\times m}\}$. Indeed, the inclusion $\mathcal{Z}' \subseteq \mathcal{Z}$ follows from $\mathrm{tr}(M^*VW^*) = \mathrm{tr}(VCW^*) = \mathrm{tr}(CW^*V) = 0$ when $W^*V = 0$ and the equality $\mathcal{Z}' = \mathcal{Z}$ follows from a dimensional argument, using for instance the rank-nullity theorem for the map $M \in \mathcal{Z}' \mapsto V^*MV \in \mathbb{K}^{m\times m}$ whose null space $\{M \in \mathbb{K}^{N\times N} : M^*V = 0\}$ has dimension $N^2 - Nm$ and whose range $\mathbb{K}^{m\times m}$ has dimension m^2. To wrap up the justification of (15), it remains to notice that, when $M^*V = VC$ and when $P = VW^*$ is a projection, we have

$$\mathrm{Re}\,\mathrm{tr}(M^*P) = \mathrm{Re}\,\mathrm{tr}(M^*VW^*) = \mathrm{Re}\,\mathrm{tr}(VCW^*) = \mathrm{Re}\,\mathrm{tr}(CW^*V) = \mathrm{Re}\,\mathrm{tr}(C).$$

Now that (15) is fully established, let us consider matrices $M \in \mathbb{K}^{N\times N}$, $\|M\|^*_{\infty\to\infty} = 1$, and $C \in \mathbb{K}^{m\times m}$, $M^*V = VC$, achieving the maximum in (15). Let us define $t \in \mathbb{R}^N_+$ with $\|t\|_2 = 1$ by

$$t_i^2 := \max_{j\in[\![1:N]\!]} |M|_{i,j}.$$

With $T := \mathrm{diag}(t)$, let us also define

$$F := V(V^*T^2V)^{-1/2} \in \mathbb{K}^{N\times m} \qquad \text{and} \qquad U := TF = TV(V^*T^2V)^{-1/2} \in \mathbb{K}^{N\times m}.$$

In fact, F and U are not well-defined if t has some zero entries, but in this case we would simply replace T by $\mathrm{diag}(t) + \varepsilon I_N$ for some $\varepsilon > 0$ that we make arbitrarily small at the end. It is readily verified that $U^*U = I_m$. Moreover, we have

$$\begin{aligned}
\lambda(\mathcal{V}_m, \ell_\infty^N) &= \mathrm{Re}\ \mathrm{tr}(C) = \mathrm{Re}\ \mathrm{tr}\left(\left(V^*T^2V\right)^{-1/2}\left(V^*T^2V\right)C\left(V^*T^2V\right)^{-1/2}\right)\\
&= \mathrm{Re}\ \mathrm{tr}\left(F^*T^2M^*F\right) = \mathrm{Re}\ \mathrm{tr}\left((TF)^*(TM^*)F\right) = \mathrm{Re}\left[\sum_{k=1}^m\sum_{i,j=1}^N (TF)^*_{k,i}(TM^*)_{i,j}F_{j,k}\right]\\
&= \mathrm{Re}\left[\sum_{k=1}^m\sum_{i,j=1}^N t_iF_{i,k}t_iM_{j,i}F_{j,k}\right] = \mathrm{Re}\left[\sum_{i,j=1}^N t_i^2M_{j,i}\sum_{k=1}^m F_{i,k}F_{j,k}\right]\\
&\le \sum_{i,j=1}^N t_i^2|M|_{j,i}|FF^*|_{i,j} \le \sum_{i,j=1}^N t_i^2t_j^2|FF^*|_{i,j} = \sum_{i,j=1}^N t_it_j|UU^*|_{i,j}.
\end{aligned}$$

Bounding the latter from above by the right-hand side of (3) before taking the maximum over all spaces $\mathcal{V}_m$ yields the upper estimate for $\lambda_{\mathbb{K}}(m, N)$.

Let us now turn to the $\ge$-part — the arguments differ from the ones of [5, 4] and they are valid in case $\mathbb{K} = \mathbb{C}$, too. Let $t \in \mathbb{R}^N_+$ with $\|t\|_2 = 1$ and $U \in \mathbb{K}^{N\times m}$ with $U^*U = I_m$ achieving the maximum in the right-hand side of (3) and let γ be the value of this maximum. We shall exploit the extremal properties of t (with U being fixed) and of U (with t being fixed) to deduce the conditions

$$\sum_{j=1}^N |UU^*|_{i,j}t_j = \gamma\, t_i \qquad \text{for all } i \in [\![1:N]\!], \tag{16}$$

$$\sum_{i,j=1}^N t_it_j|UW^*|_{i,j} \ge \sum_{i,j=1}^N t_it_j|UU^*|_{i,j} \qquad \text{for all } W \in \mathbb{K}^{N\times m} \text{ with } W^*U = I_m. \tag{17}$$

To derive the first condition, we remark that γ and t are the maximum and maximizer of $\langle |UU^*|x, x\rangle$ over all $x \in \mathbb{K}^N$ with $\|x\|_2 = 1$, so Rayleigh quotient theorem asserts that γ is the largest eigenvalue of $|UU^*|$ and t is an associated eigenvector. Condition (16) is just a rewriting of the equation $|UU^*|t = \gamma\, t$. Deriving Condition (17) requires a little more work. Given $W \in \mathbb{K}^{N\times m}$ with $W^*U = I_m$, for any $\varepsilon \in (-1, 1)$, we consider

$$U_\varepsilon := (1-\varepsilon)U + \varepsilon W.$$

An elementary calculation gives $U_\varepsilon^* U_\varepsilon = (1-\varepsilon^2) I_m + \varepsilon^2 W^* W$. Since the latter is positive definite, we can define

$$\widetilde{U}_\varepsilon := U_\varepsilon \left(U_\varepsilon^* U_\varepsilon\right)^{-1/2} \in \mathbb{K}^{N \times m}, \qquad \text{which satisfies} \quad \widetilde{U}_\varepsilon^* \widetilde{U}_\varepsilon = I_m.$$

From the truncated expansion

$$[U_\varepsilon^* U_\varepsilon]^{-1} = \left[(1-\varepsilon^2)\left(I_m + \frac{\varepsilon^2}{1-\varepsilon^2} W^* W\right)\right]^{-1} = \frac{1}{1-\varepsilon^2}\left(I_m - \frac{\varepsilon^2}{1-\varepsilon^2} W^* W + \mathcal{O}(\varepsilon^2)\right) = I_m + \mathcal{O}(\varepsilon^2),$$

we infer the truncated expansion

$$\begin{aligned}\widetilde{U}_\varepsilon \widetilde{U}_\varepsilon^* &= U_\varepsilon [U_\varepsilon^* U_\varepsilon]^{-1} U_\varepsilon^* = ((1-\varepsilon)U + \varepsilon W)\left(I_m + \mathcal{O}(\varepsilon^2)\right)((1-\varepsilon)U^* + \varepsilon W^*) \\ &= (1-\varepsilon)^2 UU^* + (1-\varepsilon)\varepsilon(UW^* + WU^*) + \mathcal{O}(\varepsilon^2) = UU^* + \varepsilon(UW^* + WU^* - 2UU^*) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

In view of the extremal property of U, we have

$$\sum_{i,j=1}^N t_i t_j |UU^*|_{i,j} \geq \sum_{i,j=1}^N t_i t_j |\widetilde{U}_\varepsilon \widetilde{U}_\varepsilon^*|_{i,j} \geq \sum_{i,j=1}^N t_i t_j \overline{\mathrm{sgn}(UU^*)_{i,j}} (\widetilde{U}_\varepsilon \widetilde{U}_\varepsilon^*)_{i,j},$$

so that

$$\sum_{i,j=1}^N t_i t_j \overline{\mathrm{sgn}(UU^*)_{i,j}} (UU^*)_{i,j} \geq \sum_{i,j=1}^N t_i t_j \overline{\mathrm{sgn}(UU^*)_{i,j}} (UU^* + \varepsilon(UW^* + WU^* - 2UU^*))_{i,j} + \mathcal{O}(\varepsilon^2)$$

For this inequality to hold whenever $\varepsilon \in (-1,1)$, the ε-term must be zero, i.e.,

$$\begin{aligned}2\sum_{i,j=1}^N t_i t_j \overline{\mathrm{sgn}(UU^*)_{i,j}} (UU^*)_{i,j} &= \sum_{i,j=1}^N t_i t_j \overline{\mathrm{sgn}(UU^*)_{i,j}} (UW^* + WU^*)_{i,j} \\ &= 2\sum_{i,j=1}^N t_i t_j \,\mathrm{Re}\left[\overline{\mathrm{sgn}(UU^*)_{i,j}} (UW^*)_{i,j}\right].\end{aligned}$$

Condition (17) is an immediate consequence of this identity. With Conditions (16) and (17) at hand, we now aim at producing a subspace with a minimal projection of norm γ. This transpires from the argument below if all the t_i's are positive by setting $\varepsilon = 0$ throughout. But we have to introduce $\varepsilon > 0$ because some of the t_i's might vanish. Thus, we consider $T' := \mathrm{diag}(t')$, where $t' \in \mathbb{R}^N$ is the ℓ_2-normalized vector with entries

$$t'_i := \sqrt{\frac{t_i^2 + \varepsilon/N}{1+\varepsilon}} > 0, \qquad i \in [\![1:N]\!].$$

We also introduce

$$P := (T')^{-1} UU^* T' \in \mathbb{K}^{N \times N},$$

which is (the matrix of) a projection by virtue of $P^2 = P$. It is not complicated to see that any other projection onto the range of P takes the form $Q = (T')^{-1}UW^*T'$ for some $W \in \mathbb{K}^{N\times m}$ satisfying $W^*U = I_m$. We observe that

$$\begin{aligned}\|Q\|_{\infty\to\infty} &= \max_{i\in[\![1:N]\!]} \sum_{j=1}^N |(T')^{-1}UW^*T'|_{i,j} = \max_{i\in[\![1:N]\!]} \sum_{j=1}^N \frac{1}{t'_i}|UW^*|_{i,j}t'_j \geq \sum_{i=1}^N (t'_i)^2 \sum_{j=1}^N \frac{1}{t'_i}|UW^*|_{i,j}t'_j \\ &= \sum_{i,j=1}^N t'_it'_j|UW^*|_{i,j} \geq \frac{1}{1+\varepsilon}\sum_{i,j=1}^N t_it_j|UW^*|_{i,j} \underset{(17)}{\geq} \frac{1}{1+\varepsilon}\sum_{i,j=1}^N t_it_j|UU^*|_{i,j} \underset{(16)}{=} \frac{\gamma}{1+\varepsilon}.\end{aligned}$$

In the case $\varepsilon = 0$, we stress that equality holds all the way through when $W = U$ thanks to (16), meaning that P is a minimal projection onto its range. In the case $\varepsilon > 0$, we can still bound $\lambda_{\mathbb{K}}(m, N)$ from below as

$$\lambda_{\mathbb{K}}(m,N) \geq \lambda(\mathrm{ran}(P), \ell_\infty^N) \geq \frac{\gamma}{1+\varepsilon}.$$

The desired result is obtained by taking $\varepsilon > 0$ arbitrarily small. $\square$

Appendix B: Computed Maximal and Quasimaximal Relative Projection Constants

We have applied the computational procedures described in Section 3 to determine the values of $\mu_{\mathbb{R}}(m, N)$ and $\lambda_{\mathbb{R}}(m, N)$ for $1 \leq m \leq N \leq 10$. They are displayed in Tables 1 and 2 below — strictly speaking, the values in Table 2 are only guaranteed to be lower bounds for $\lambda_{\mathbb{R}}(m, N)$. Optimal spaces $\mathcal{V}_m$ in (2), optimal vectors t in (3)-(4), optimal matrices U in (3)-(6), and optimal Seidel matrices in (4)-(7) have also been determined. They are all included in the MATLAB reproducible, where the reduced quantities $\widetilde{\mu}_{\mathbb{R}}(m, N)$ and $\widetilde{\lambda}_{\mathbb{R}}(m, N)$ are also available for $1 \leq m \leq N \leq 10$.

Table 1: Values of $\mu_{\mathbb{R}}(m, N)$

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
$N=1$	1.0000									
$N=2$	1.0000	1.0000								
$N=3$	1.0000	1.3333	1.0000							
$N=4$	1.0000	1.3090	1.5000	1.0000						
$N=5$	1.0000	1.3123	1.5123	1.6000	1.0000					
$N=6$	1.0000	1.3333	1.6180	1.6667	1.6667	1.0000				
$N=7$	1.0000	1.3250	1.5962	1.7391	1.7536	1.7143	1.0000			
$N=8$	1.0000	1.3257	1.5875	1.7892	1.8375	1.8257	1.7500	1.0000		
$N=9$	1.0000	1.3333	1.5889	1.8020	1.9132	1.9223	1.8889	1.7778	1.0000	
$N=10$	1.0000	1.3292	1.5936	1.8485	2.0000	2.0485	1.9936	1.9292	1.8000	1.0000

Table 2: Values of (lower bounds for) $\lambda_{\mathbb{R}}(m,N)$

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
$N=1$	1.0000									
$N=2$	1.0000	1.0000								
$N=3$	1.0000	1.3333	1.0000							
$N=4$	1.0000	1.3333	1.5000	1.0000						
$N=5$	1.0000	1.3333	1.5224	1.6000	1.0000					
$N=6$	1.0000	1.3333	1.6180	1.6667	1.6667	1.0000				
$N=7$	1.0000	1.3333	1.6180	1.7398	1.7536	1.7143	1.0000			
$N=8$	1.0000	1.3333	1.6180	1.7923	1.8377	1.8257	1.7500	1.0000		
$N=9$	1.0000	1.3333	1.6180	1.8050	1.9142	1.9223	1.8889	1.7778	1.0000	
$N=10$	1.0000	1.3333	1.6180	1.8496	2.0000	2.0486	1.9939	1.9294	1.8000	1.0000

Appendix C: Equality of Maximal and Quasimaximal Relative Projection Constants

As pointed out in Remark 7, the quasimaximal relative projection constant $\mu_{\mathbb{K}}(m,N)$ and the maximal relative projection constant $\lambda_{\mathbb{K}}(m,N)$ are equal when there is an equiangular tight frame of N vectors in $\mathbb{K}^m$, but this is not the only case. If equality occurs, then something can be said about the minimal projections onto the optimal spaces.

Proposition 13. Given $N \geq m$, if $\mu_{\mathbb{K}}(m,N) = \lambda_{\mathbb{K}}(m,N)$, then there exists an m-dimensional subspace $\mathcal{V}_m$ of ℓ_∞^N such that $\lambda(\mathcal{V}_m, \ell_\infty^N) = \lambda_{\mathbb{K}}(m,N)$ for which the orthogonal projection onto $\mathcal{V}_m$ is a minimal projection.

Proof. The equality between $\mu_{\mathbb{K}}(m,N)$ and $\lambda_{\mathbb{K}}(m,N)$ tells us that $t = [1,\ldots,1]^\top/\sqrt{N}$ is extremal in (3). Let us denote by $U \in \mathbb{K}^{N\times m}$ the matrix achieving the maximum in (3) and let us consider the orthogonal projection $P := UU^*$. Following the argument used in Appendix A to prove the $\geq$-part of (3), we can say that the range of P is an m-dimensional space achieving the maximum in (2) and that P is a minimal projection. $\square$

We believe that the converse to Proposition 13 holds, but we were unsuccessful in proving it.

Appendix D: Proof of the Bound (11) on the Maximal Relative Projection Constant

We show here how the expression (4) can be exploited to provide a novel justification of the upper bound (11) originally derived in Theorem 1 of [12] and unimproved since then, namely

$$\lambda_{\mathbb{K}}(m,N) \leq \frac{m}{N}\left(1 + \sqrt{\frac{(N-1)(N-m)}{m}}\right) = \frac{m}{N} + \frac{\sqrt{(N-1)m(N-m)}}{N}. \tag{18}$$

We start by establishing the following simple result.

Lemma 14. Given integers $N \geq m$, if $x \in \mathbb{R}^N$ satisfies $\sum_{k=1}^N x_k = 0$ and $\sum_{k=1}^N x_k^2 = 1$, then

$$\sum_{k=1}^m x_k \leq \sqrt{\frac{m(N-m)}{N}}.$$

Proof. Consider the vector $\mathbf{1} := [1, \ldots, 1]^\top \in \mathbb{R}^N$ whose entries are all equal to one, as well as the vector $\mathbf{1}_m := [1, \ldots, 1, 0, \ldots, 0]^\top \in \mathbb{R}^N$ whose first m entries are equal to one and whose remaining entries are all equal to zero. The task at hand is to maximize $\langle \mathbf{1}_m, x \rangle$ subject to $\langle \mathbf{1}, x \rangle = 0$ and $\|x\|_2 = 1$, or equivalently to minimize $\|\mathbf{1}_m - x\|_2$ for x in the unit sphere of the hyperplane $\mathcal{H} := \mathbf{1}^\perp$. As a picture would reveal, the extremizer $x^\star$ is the ℓ_2-normalized orthogonal projection of $\mathbf{1}_m$ onto $\mathcal{H}$, i.e., $x^\star = P_{\mathcal{H}}(\mathbf{1}_m)/\|P_{\mathcal{H}}(\mathbf{1}_m)\|_2$. Then the value of the maximum of $\langle \mathbf{1}_m, x \rangle$ is

$$\langle \mathbf{1}_m, x^\star \rangle = \frac{\langle \mathbf{1}_m, P_{\mathcal{H}}(\mathbf{1}_m) \rangle}{\|P_{\mathcal{H}}(\mathbf{1}_m)\|_2} = \|P_{\mathcal{H}}(\mathbf{1}_m)\|_2 = \sqrt{\|\mathbf{1}_m\|_2^2 - \langle \mathbf{1}_m, \mathbf{1} \rangle^2 / \|\mathbf{1}\|_2^2} = \sqrt{m - m^2/N},$$

which is indeed the announced bound for $\langle \mathbf{1}_m, x \rangle$. $\square$

Turning now to the justification of (18), given $T = \operatorname{diag}(t)$ with $t \in \mathbb{R}_+^N$ and $\|t\|_2 = 1$ and given $A = I_N + B$ with $B \in \mathcal{S}_{\mathcal{K}}^{N \times N}$, we set

$$x_k = \sqrt{\frac{N}{N-1}} \left(\lambda_k^\downarrow(TAT) - \frac{1}{N} \right), \qquad k \in [\![1 : N]\!].$$

The condition $\sum x_k = 0$ holds because $\sum \lambda_k^\downarrow(TAT) = \operatorname{tr}(TAT) = 1$, while the condition $\sum x_k^2 = 1$ holds because

$$\begin{aligned} \sum_{k=1}^N \left(\lambda_k^\downarrow(TAT) - \frac{1}{N} \right)^2 &= \sum_{k=1}^N \lambda_k^\downarrow(TAT)^2 - \frac{2}{N} \sum_{i=k}^N \lambda_k^\downarrow(TAT) + \frac{1}{N} = \|TAT\|_F^2 - \frac{2}{N} \operatorname{tr}(TAT) + \frac{1}{N} \\ &= 1 - \frac{2}{N} + \frac{1}{N} = \frac{N-1}{N}. \end{aligned}$$

Thus, applying Lemma 14 gives

$$\sum_{k=1}^m \sqrt{\frac{N}{N-1}} \left(\lambda_k^\downarrow(TAT) - \frac{1}{N} \right) \leq \sqrt{\frac{m(N-m)}{N}}, \quad \text{i.e.,} \ \sum_{k=1}^m \lambda_k^\downarrow(TAT) \leq \frac{m}{N} + \frac{\sqrt{(N-1)m(N-m)}}{N}.$$

Taking the maximum over t and A yields the upper bound on $\lambda_{\mathbb{K}}(m, N)$ announced in (18). $\square$