Open questions around the spline orthoprojector

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Abstract

An open problem about the max-norm of the spline orthoprojector is formulated. It gives rise to several conjectures about the orthoprojectors onto some spaces of polynomials.

We discuss in this note several open questions concerning the orthogonal projector onto spline spaces. The main advance on the subject has of course been Shadrin's proof [9] of de Boor's conjecture [1] that the max-norm of the orthoprojector is bounded independently of the breakpoint sequence underlying the spline space. To give a more precise statement, we shall consider a breakpoint sequence $\Delta = (-1 = t_0 < t_1 < \cdots < t_N < t_{N+1} = 1)$, and introduce the space

$$\mathcal{S}_{k,m}(\Delta) := \left\{ s \in \mathcal{C}^{m-1}[-1,1] : s_{|(t_{i-1},t_i)} \text{ is a polynomial of degree } < k,\ i = 1, \ldots, N+1 \right\}$$

of splines of order k satisfying m smoothness conditions at each interior breakpoint of Δ. Notice that we have to require $0 \le m \le k-1$. Following the practice of writing P_V for the orthogonal projector onto a space V, the orthoprojector from $L_2[-1,1]$ onto $\mathcal{S}_{k,m}(\Delta)$ is denoted by $P_{\mathcal{S}_{k,m}(\Delta)}$. We shall regard this projector as an operator on $L_\infty[-1,1]$, then consider its associated norm, and finally examine the supremum over all breakpoint sequences. In short, we are interested in the quantity

$$\Lambda_{k,m} := \sup_{\Delta} \left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty = \sup_{\Delta} \sup_{\|f\|_\infty \le 1} \left\| P_{\mathcal{S}_{k,m}(\Delta)}(f) \right\|_\infty .$$

Shadrin's theorem ensures that

$$\Lambda_{k,m} \le \Lambda_{k,k-1} < \infty.$$

But estimating the quantity $\Lambda_{k,m}$, or merely its order, remains a challenge — only for $m = 1, 2$ is it known [2] that the ratio $\Lambda_{k,m}/\sqrt{k}$ is bounded from above and from below. We are going to suggest some open problems that go in the direction of a precise estimate. Significantly, it is our belief that the lower bound $\sigma_{k,m}$ of [2] is the actual value of $\Lambda_{k,m}$. This belief incorporates Shadrin's conjecture [9] that $\Lambda_{k,k-1} = 2k-1$.

1 The spline orthoprojector question

Our main conjecture takes an elegant form. It reads

$$\Lambda_{k,m} := \sup_{\Delta} \left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty \stackrel{?}{=} \frac{k}{k-m} \left\| P_{\mathcal{P}_{k,m}} \right\|_\infty,$$

where the subspace $\mathcal{P}_{k,m}$ of $\mathcal{C}[-1,1]$ is made of polynomials of order k that vanish m-fold at the endpoint -1, that is

$$\mathcal{P}_{k,m} := \operatorname{span}\Big[(1+\bullet)^m, \ldots, (1+\bullet)^{k-1}\Big].$$

This presentation, however, hides the important details that follow. We shall use the subscript N to indicate that the breakpoint sequence $\Delta_N := (-1 = t_0 < t_1 < \cdots < t_N < t_{N+1} = 1)$ is required to possess N interior breakpoints. With the linear functional $\mathcal{E}_x$ denoting the evaluation at the point x, we consider the quantities

$$\begin{aligned}
\Lambda_{k,m,N} &:= \sup_{\Delta_N} \left\| P_{\mathcal{S}_{k,m}(\Delta_N)} \right\|_\infty, \\
\Upsilon_{k,m,N} &:= \sup_{\Delta_N} \left\| \mathcal{E}_1 P_{\mathcal{S}_{k,m}(\Delta_N)} \right\|_\infty = \sup_{\Delta_N} \sup_{\|f\|_\infty \le 1} \left| P_{\mathcal{S}_{k,m}(\Delta_N)}(f)(1) \right|.
\end{aligned}$$

Beware that this $\Upsilon_{k,m,N}$ corresponds to the $\Upsilon_{k,m,N+1}$ of [2]. These quantities are connected with quantities relative to the space $\mathcal{P}_{k,m}$, namely

$$\rho_{k,m} := \varrho_{k,m}(1), \qquad \text{where } \varrho_{k,m}(x) := \left\| \mathcal{E}_x P_{\mathcal{P}_{k,m}} \right\|_\infty = \sup_{\|f\|_\infty \le 1} \left| P_{\mathcal{P}_{k,m}}(f)(x) \right|,$$

$$\sigma_{k,m} := \frac{k}{k-m}\, \rho_{k,m}.$$

Indeed, we established in [2] that

$$\Upsilon_{k,m,N} \ge \frac{m}{k}\, \Upsilon_{k,m,N-1} + \rho_{k,m}, \qquad \text{i.e.} \quad \big[\Upsilon_{k,m,N} - \sigma_{k,m}\big] \ge \frac{m}{k} \big[\Upsilon_{k,m,N-1} - \sigma_{k,m}\big].$$

Since the observation $\Upsilon_{k,m,0} = \rho_{k,0} = \sigma_{k,0}$ is immediate, we derived that

$$\Upsilon_{k,m,N} \ge \left[\left(\frac{m}{k}\right)^N\right] \sigma_{k,0} + \left[1 - \left(\frac{m}{k}\right)^N\right] \sigma_{k,m}. \tag{1}$$

We take $\Lambda_{k,m} \geq \Lambda_{k,m,N} \geq \Upsilon_{k,m,N}$ into account, and we retrieve the lower bound $\sigma_{k,m}$ previously mentioned by letting N grow to infinity. Namely, we have

$$(2) \qquad \Lambda_{k,m} \geq \sigma_{k,m}, \qquad \text{that is} \quad \sup_{\Delta} \left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty \geq \frac{k}{k-m} \left\| \mathcal{E}_1 P_{\mathcal{P}_{k,m}} \right\|_\infty.$$

To match our main conjecture, we believe that equality occurs in (2), and additionally in (1).

Problem 1. $\Lambda_{k,m,N} \stackrel{?}{=} \Upsilon_{k,m,N} \stackrel{?}{=} \left[\left(\frac{m}{k} \right)^N \right] \sigma_{k,0} + \left[1 - \left(\frac{m}{k} \right)^N \right] \sigma_{k,m}.$

The first nontrivial case, $k = 2$ and $m = 1$, corresponds to continuous broken lines. It was settled by Malyugin in the rather technical paper [7]. Let us point out that the problem breaks into two distinct parts. The first one asks if the Lebesgue function $x \mapsto \sup_\Delta \|\mathcal{E}_x P_{\mathcal{S}_{k,m}(\Delta)}\|_\infty$ achieves its maximum at the endpoints of the interval $[-1,1]$ — Section 2 contains related questions. The other one is concerned with the optimal breakpoint sequences that would yield equality in $\Upsilon_{k,m,N} \geq \frac{m}{k} \Upsilon_{k,m,N-1} + \rho_{k,m}$ — Section 3 investigates a particular case.

2 Purely polynomial questions

Assuming a positive answer to Problem 1, we could deduce our main conjecture using the two further ingredients

$$\left\| P_{\mathcal{P}_{k,m}} \right\|_\infty \stackrel{?}{=} \left\| \mathcal{E}_1 P_{\mathcal{P}_{k,m}} \right\|_\infty \qquad \text{and} \qquad \sigma_{k,m} \stackrel{?}{\geq} \sigma_{k,0}.$$

These issues are to be explored in the next two subsections.

2.1 The maximization of the Lebesgue function of $P_{\mathcal{P}_{k,m}}$

We are confident that the max-norm of the orthoprojector $P_{\mathcal{P}_{k,m}}$ is taken at the right endpoint of the interval $[-1,1]$. To rephrase this statement, we expect a positive answer to the following problem.

Problem 2. $\sup_{x \in [-1,1]} \varrho_{k,m}(x) \stackrel{?}{=} \varrho_{k,m}(1).$

We could verify this numerically for $k \le 20$ by sampling the functions $\varrho_{k,m}$ at 201 equidistant points. The problem can also be solved for some particular values of m. For instance, the solution is almost immediate for $m = k-1$, but it appears tedious for $m = k-2$, hence we chose not to include it. For $m = 0$, the statement can be found in [6], and can be traced further back to [4, p.326–327]. The method of proof — the only one we are aware of — goes along the lines below. Specializing a result [3] about Jacobi polynomials to the Legendre polynomials P_n, there exists a symmetric kernel $K(x,y,z) \ge 0$ such that

$$P_n(x)P_n(y) = \int_{-1}^{1} P_n(z)\, K(x,y,z)\, dz, \qquad x,y \in [-1,1]. \tag{3}$$

Then, in view of the expression

$$P_{\mathcal{P}_{k,0}}(f)(x) = \sum_{i=0}^{k-1} \frac{\langle P_i, f\rangle}{\|P_i\|_2^2}\, P_i(x) =: \int_{-1}^{1} \sum_{i=0}^{k-1} \frac{P_i(x)P_i(y)}{h_i} \;\cdot\; f(y)\, dy =: \int_{-1}^{1} \mathcal{K}_{k,0}(x,y) \;\cdot\; f(y)\, dy,$$

we note that the max-norm of the functional $\mathcal{E}_x P_{\mathcal{P}_{k,0}}$ equals the L_1-norm of the polynomial $\mathcal{K}_{k,0}(x,\bullet)$. The latter satisfies

$$\mathcal{K}_{k,0}(x,y) = \sum_{i=0}^{k-1} \frac{P_i(x)P_i(y)}{h_i} = \sum_{i=0}^{k-1} \frac{1}{h_i} \int_{-1}^{1} P_i(z)\, K(x,y,z)\, dz = \int_{-1}^{1} \mathcal{K}_{k,0}(z,1)\, K(x,y,z)\, dz.$$

We have made use of the fact that $P_i(1) = 1$ in the last equality. We now use $(3)_{n=0}$, that is $\int_{-1}^{1} K(x,y,z)\, dz = 1$, and the symmetry of $K(x,y,z)$ to derive

$$\begin{aligned}
\big\|\mathcal{E}_x P_{\mathcal{P}_{k,0}}\big\|_\infty &= \int_{-1}^{1} \Big|\mathcal{K}_{k,0}(x,y)\Big|\, dy = \int_{-1}^{1} \Big| \int_{-1}^{1} \mathcal{K}_{k,0}(z,1)\, K(x,y,z)\, dz \Big|\, dy \\
&\le \int_{-1}^{1}\int_{-1}^{1} \Big|\mathcal{K}_{k,0}(z,1)\Big|\, K(x,y,z)\, dz\, dy = \int_{-1}^{1} \Big|\mathcal{K}_{k,0}(z,1)\Big| \int_{-1}^{1} K(x,y,z)\, dy\; dz \\
&= \int_{-1}^{1} \Big|\mathcal{K}_{k,0}(z,1)\Big|\, dz = \big\|\mathcal{E}_1 P_{\mathcal{P}_{k,0}}\big\|_\infty.
\end{aligned}$$

It is tempting modify this argument by replacing the orthogonal basis $(P_0,\ldots,P_{k-1})$ of $\mathcal{P}_{k,0}$ by the orthogonal bases $(p_0,\ldots,p_{k-1-m})$ or $(q_m,\ldots,q_{k-1})$ of $\mathcal{P}_{k,m}$, where

$$p_i(x) := \Big(\frac{1+x}{2}\Big)^m P_i^{(0,2m)}(x) \qquad \text{and} \qquad q_i(x) := \Big(\frac{1+x}{2}\Big)^i P_{k-1-i}^{(0,2i+1)}(x).$$

Here we have used the traditional notation $P_n^{(\alpha,\beta)}(x)$ for the Jacobi polynomials of degree n, which are orthogonal on $[-1,1]$ with respect to the weight $(1-x)^\alpha(1+x)^\beta$ and are normalized by $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. Unfortunately, such a program is doomed. Indeed, the existence of a kernel $K(x,y,z) \geq 0$ such that

$$f_i(x) f_i(y) = \int_{-1}^{1} f_i(z)\, K(x,y,z)\, dz$$

for some functions $f_0, \ldots, f_n$ defined on $[-1,1]$ is equivalent to the property

$$\sum_{i=0}^{n} a_i f_i(z) \geq 0, \quad z \in [-1,1] \qquad \Longrightarrow \qquad \sum_{i=0}^{n} a_i f_i(x) f_i(y) \geq 0, \quad x,y \in [-1,1].$$

In the case $k = 3$ and $m = 1$, the choices $(a_0, a_1) = (1, -1)$ for (p_0, p_1) and $(a_0, a_1) = (-1, 1)$ for (q_1, q_2) reveal — graphically or not — that this property does not hold.

Note that a reformulation of Problem 2 in terms of the L_1-norm of a polynomial depending on a parameter x, and its speculative maximization at $x = 1$, is not specific to the case $m = 0$. Precisely, with $n := k - 1 - m$, the kernel of the orthoprojector $P_{\mathcal{P}_{k,m}}$ is

$$\mathcal{K}_{k,m}(x,y) := \sum_{i=0}^{n} \frac{p_i(x)\, p_i(y)}{\|p_i\|_2^2} = \sum_{i=0}^{n} \frac{(1+x)^m (1+y)^m}{h_i^{(0,2m)}}\, P_i^{(0,2m)}(x) P_i^{(0,2m)}(y),$$

and Problem 2 asks if $\left\|\mathcal{K}_{k,m}(x,\bullet)\right\|_1$ is maximized at $x = 1$. We put forward a stronger question.

Problem 3. $\left\|\mathcal{K}_{k,m}(x,\bullet)\right\|_p \overset{?}{\leq} \left\|\mathcal{K}_{k,m}(1,\bullet)\right\|_p$ for all $p \in [1,\infty]$.

This is merely motivated by the results for $p = 2$ and $p = \infty$. Both are consequences of the fact — see [10, Theorem 7.2, p.161] — that $\max\limits_{x \in [-1,1]} |p_i(x)| = p_i(1)$. For $p = 2$, we simply write

$$\left\|\mathcal{K}_{k,m}(x,\bullet)\right\|_2^2 = \sum_{i=0}^{n} \frac{p_i(x)^2}{\|p_i\|_2^2} \leq \sum_{i=0}^{n} \frac{p_i(1)^2}{\|p_i\|_2^2} = \left\|\mathcal{K}_{k,m}(1,\bullet)\right\|_2^2 = \frac{k^2 - m^2}{2}.$$

To obtain the last equality, we have applied the linear functional $\mathcal{E}_1 P_{\mathcal{P}_{k,m}}$ to the polynomial $\mathcal{K}_{k,m}(1,\bullet) \in \mathcal{P}_{k,m}$ to get

$$\mathcal{K}_{k,m}(1,1) = \left\|\mathcal{K}_{k,m}(1,\bullet)\right\|_2^2,$$

and we read the value of $\mathcal{K}_{k,m}(1,1)$ in the special form — see [10, p.71] —

$$\mathcal{K}_{k,m}(1,y) = \frac{k+m}{2} \left(\frac{1+y}{2}\right)^m P_{k-1-m}^{(1,2m)}(y). \tag{4}$$

The case $p = \infty$ is just as simple to deal with, since

$$\left\|\mathcal{K}_{k,m}(x,\bullet)\right\|_\infty \leq \sum_{i=0}^{n} \frac{|p_i(x)|\, \|p_i\|_\infty}{\|p_i\|_2^2} \leq \sum_{i=0}^{n} \frac{p_i(1)\, p_i(1)}{\|p_i\|_2^2} = \mathcal{K}_{k,m}(1,1) = \left\|\mathcal{K}_{k,m}(1,\bullet)\right\|_\infty = \frac{k^2 - m^2}{2}.$$

2.2 The behavior of the constant $\sigma_{k,m}$

2.2.1 Monotonicity

Identity (4) offers an easy way to compute the quantities $\rho_{k,m}$, and in turn $\sigma_{k,m}$. Backed up by a table of values for $k \le 50$, we trust that the next problem has a positive answer.

Problem 4. Is $\sigma_{k,m}$ increasing with m ?

It is worth remembering that $\sigma_{k,m}$ is the conjectured value for $\Lambda_{k,m} := \sup_\Delta \left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty$. Note that, even though the inequality $\Lambda_{k,m} \le \Lambda_{k,k-1}$ is known, the inequality $\sigma_{k,m} \le \sigma_{k,k-1}$ is not. Incidentally, neither is the inequality $\sigma_{k,0} \le \sigma_{k,m}$. In proving the monotonicity of $\sigma_{k,m}$, an elementary step should surely be the identity

(5) $$\mathcal{K}_{k,m}(x,y) = \mathcal{K}_{k,m+1}(x,y) + \frac{2m+1}{2}\, q_m(x)\, q_m(y),$$

derived from the orthogonal decomposition

$$\mathcal{P}_{k,m} = \mathcal{P}_{k,m+1} \overset{\perp}{\oplus} \operatorname{span}\big[q_m\big], \qquad q_m(x) := \Big(\frac{1+x}{2}\Big)^m P_{k-1-m}^{(0,2m+1)}(x).$$

This identity was exploited in [2] to establish that $\sigma_{k,0} \le \sigma_{k,1}$. The argument relies on the symmetry relation for Jacobi polynomials, that is $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, and reads

$$\begin{aligned}
\sigma_{k,0} = \rho_{k,0} &= \int_{-1}^{1} \big|\mathcal{K}_{k,0}(1,y)\big|\, dy = \int_{-1}^{1} \big|\mathcal{K}_{k,1}(1,y) + \frac{1}{2} P_{k-1}^{(0,1)}(y)\big|\, dy \\
&\le \int_{-1}^{1} \big|\mathcal{K}_{k,1}(1,y)\big|\, dy + \frac{1}{2}\int_{-1}^{1} \big|P_{k-1}^{(0,1)}(y)\big|\, dy = \int_{-1}^{1} \big|\mathcal{K}_{k,1}(1,y)\big|\, dy + \frac{1}{k}\int_{-1}^{1} \frac{k}{2}\big|P_{k-1}^{(1,0)}(y)\big|\, dy \\
&= \rho_{k,1} + \frac{1}{k}\,\rho_{k,0} = \frac{k-1}{k}\,\sigma_{k,1} + \frac{1}{k}\,\sigma_{k,0}.
\end{aligned}$$

The inequality $\sigma_{k,0} \le \sigma_{k,1}$ follows. This simple argument based on a triangle inequality cannot be carried over to all m, not even to prove that $\sigma_{k,0} \le \sigma_{k,m}$, as a numerical experiment with $k = 6$ and $m = 3$ would reveal.

2.2.2 Order of growth

The order of growth of the quantity $\sigma_{k,m}$, or equivalently of $\rho_{k,m}$, constitutes another critical issue. We believe that the max-norm of the orthoprojector onto the space $\mathcal{P}_{k,m}$ is bounded from above and from below by a constant times the square root of the dimension of the space.

Problem 5. $\rho_{k,m} \overset{?}{\asymp} \sqrt{k-m}$.

Some particular evaluations of $\rho_{k,m}$ were given in [2]. They confirm the previous guess when $m = k-3, k-2, k-1$ and when m is independent of k. We can also add the claim that $\rho_{k,m} \asymp \sqrt{k}$ when $m \leq \mathrm{const} \cdot \sqrt{k}$ with $\mathrm{const} < 2\sqrt{2/\pi}$. Indeed, we can deduce from (5) that

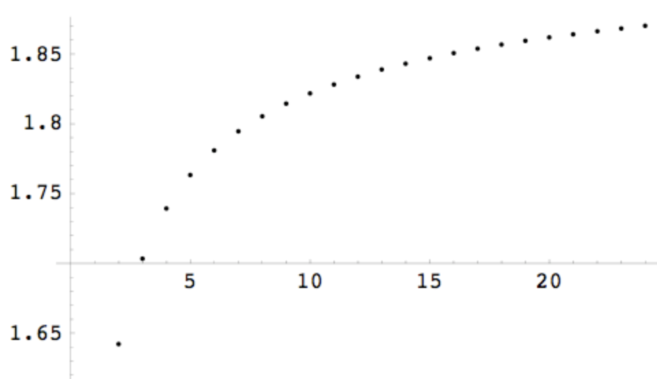
$$\mathcal{K}_{k,0}(x,y) = \mathcal{K}_{k,m}(x,y) + \sum_{i=0}^{m-1} \frac{2i+1}{2}\, q_i(x)\, q_i(y) =: \mathcal{K}_{k,m}(x,y) + \mathcal{H}_{k,m}(x,y),$$

where $\mathcal{H}_{k,m}(x,y)$ denotes the kernel associated with the orthogonal projector onto $\mathcal{P}_{k,0} \overset{\perp}{\ominus} \mathcal{P}_{k,m}$. We then have

$$\begin{aligned} \Big| \big\|\mathcal{K}_{k,m}(1,\bullet)\big\|_1 - \big\|\mathcal{K}_{k,0}(1,\bullet)\big\|_1 \Big| &\leq \big\|\mathcal{H}_{k,m}(1,\bullet)\big\|_1 \leq \sqrt{2}\,\big\|\mathcal{H}_{k,m}(1,\bullet)\big\|_2 \\ &= \sqrt{2}\left[\big\|\mathcal{K}_{k,0}(1,\bullet)\big\|_2^2 - \big\|\mathcal{K}_{k,m}(1,\bullet)\big\|_2^2 \right]^{1/2} = \sqrt{2}\left[\frac{k^2}{2} - \frac{k^2-m^2}{2} \right]^{1/2} = m. \end{aligned}$$

The claim is simply derived from the behavior $\big\|\mathcal{K}_{k,0}(1,\bullet)\big\|_1 \underset{k\infty}{\sim} 2\sqrt{2/\pi}\,\sqrt{k}$, recalled in [2].

Let us point out that, to unite $\rho_{k,0} \asymp \sqrt{k}$ and $\rho_{k,k-1} \asymp 1$, the order $k^{(k-m)/(2k)}$ could also seem natural. However, it would impose the order $k^{1/4}$ for $\rho_{k,k/2}$, whereas the predicted order $k^{1/2}$ is much more plausible. We illustrate this with the graph of $\rho_{2m,m}/\sqrt{m}$ against m.



Remark. The table of values also suggests that $\rho_{k,m}$ increases with k. This has been shown by Qu and Wong in [8] for $m = 0$. Their technique — use of an asymptotic expansion for k large enough, numerical evaluations otherwise — cannot be adapted here.

2.2.3 Monotonicity and order of growth

The previous two problems could be tackled at once, provided this next one could be solved. It has been verified numerically for $k \leq 50$. Moreover, it can be checked for $m = k-3$ and $m = k-2$ using the precise values of $\rho_{k,k-3}$, $\rho_{k,k-2}$, and $\rho_{k,k-1}$.

Problem 6. $\rho_{k,m} \overset{?}{\leq} \dfrac{k-m-1/2}{k-m-1}\, \rho_{k,m+1}.$

Observe that the monotonicity of $\sigma_{k,m}$ simply follows from $k-m-1/2 \leq k-m$. Finding the order $\sqrt{k-m}$ requires a little more work. First, we apply the above speculative inequality repeatedly to obtain

$$\rho_{k,m} \leq \frac{(k-m-1/2)\cdots(3/2)}{(k-m-1)\ \cdots\ 1}\, \rho_{k,k-1} = \frac{k-m}{1/2}\binom{k-m-1/2}{k-m}\rho_{k,k-1},$$

$$\rho_{k,m} \geq \frac{(k-m)\quad\cdots\ (k-1)}{(k-m+1/2)\cdots(k-1/2)}\, \rho_{k,0} = \frac{k-m}{k}\,\frac{\binom{k-m-1/2}{k-m}}{\binom{k-1/2}{k}}\,\rho_{k,0}.$$

We then make use of $\rho_{k,k-1} = 2 - 1/k \leq 2$ and of $\rho_{k,0} \geq 2k\binom{k-1/2}{k}$, which is obtained from

$$\left\| \mathcal{K}_{k,0}(1,\bullet) \,/\, \mathrm{ldg}[\mathcal{K}_{k,0}(1,\bullet)] \right\|_1 \geq \left\| U_{k-1} \,/\, \mathrm{ldg}[U_{k-1}] \right\|_1, \quad U_{k-1} : \text{second-kind Chebyshev polynomial,}$$

where the notation $\mathrm{ldg}[p]$ stands for the leading coefficient of a polynomial p. Assuming that Problem 6 has a positive answer, we would therefore have

$$2\,(k-m)\binom{k-m-1/2}{k-m} \;\leq\; \rho_{k,m} \;\leq\; 4\,(k-m)\binom{k-m-1/2}{k-m}.$$

Finally, Wallis's inequality

$$\frac{1}{\sqrt{\pi(n+1/2)}} \leq \binom{n-1/2}{n} = \frac{1}{2^{2n}}\binom{2n}{n} \leq \frac{1}{\sqrt{\pi n}}$$

would confirm the order $\sqrt{k-m}$ predicted for the quantity $\rho_{k,m}$.

3 The optimality question in a particular case

In view of Problem 1, the inequality $\Upsilon_{k,m,N} \geq \dfrac{m}{k}\,\Upsilon_{k,m,N-1} + \rho_{k,m}$ should be sharp. Let us recall the way it was obtained. A breakpoint sequence $\Delta_{N-1} = (-1 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1)$

was refined to form

$$\Delta_t = (-1 = t_0 < t_1 < \cdots < t_{N-1} < t < t_N = 1).$$

We then established that

$$\Upsilon_{k,m,N} \geq \big\|\mathcal{E}_1 P_{\mathcal{S}_{k,m}(\Delta_t)}\big\| \geq \frac{m}{k}\,\big\|\mathcal{E}_1 P_{\mathcal{S}_{k,m}(\Delta_{N-1})}\big\| + \big\|\mathcal{E}_1 P_{\mathcal{P}_{k,m}}\big\| - \varepsilon_t,$$

with $\varepsilon_t \underset{t\to 1}{\longrightarrow} 0$. We concluded by letting t tend to 1, and by taking the supremum over Δ_{N-1}.

Since we believe this situation to be extremal, we expect e.g. to get

$$\sup_{\Delta_N} \big\|P_{\mathcal{S}_{k,m}(\Delta_N)}\big\|_\infty \overset{?}{=} \lim_{\theta\to 0} \big\|P_{\mathcal{S}_{k,m}(\Delta_N(\theta))}\big\|_\infty, \quad \text{where} \quad \Delta_N(\theta) := (-1 < 1-\theta < \cdots < 1-\theta^N < 1).$$

In any event, if the breakpoint sequence $\delta = (-1 < t < 1)$ is prescribed to only possess one interior breakpoint, the extremal situation should occur when the interior breakpoint approaches an endpoint, say when $t \to 1$. We are now going to examine this particular case for continuous splines, that is for $m = 1$. Our objective is to show that the problem is equivalent to yet another polynomial question, which is reminiscent of Problem 2. It has been verified numerically for $k \leq 25$ by sampling at 201 equidistant points.

Problem 7. Is $\displaystyle\int_{-1}^{1} \Big[\big|\mathcal{K}_{k,0}(u,v) - \mathcal{K}_{k,1}(u,v)\big| + \big|\mathcal{K}_{k,1}(u,v)\big|\Big]\, dv$ maximized at $u = 1$?

Let us justify our equivalence claim. In terms of the Lebesgue function

$$L_t(x) := \sup_{\|f\|_\infty \leq 1} \big|P_{\mathcal{S}_{k,1}(\delta)}(f)(x)\big|, \qquad x \in [-1,1],$$

the question of the optimal breakpoint sequence is combined with Problem 1 to give

$$L_t(x) \overset{?}{\leq} \lim_{\tau\to 1} L_\tau(1) \overset{?}{=} \Big[\frac{1}{k}\Big]\sigma_{k,0} + \Big[1 - \frac{1}{k}\Big]\sigma_{k,1}. \tag{6}$$

This should hold for any x, which by symmetry can be assumed to belong to the interval $[t,1]$. According to the method used in [2], the orthogonal complement of $\mathcal{S}_{k,1}(\delta)$ in $\mathcal{S}_{k,0}(\delta)$ can be described explicitly. We have

$$\mathcal{S}_{k,0}(\delta) = \mathcal{S}_{k,1}(\delta) \overset{\perp}{\oplus} \operatorname{span}\big[s_t\big], \qquad s_t(y) := \begin{cases} \dfrac{2}{t+1}\, P_{k-1}^{(1,0)}\Big(\dfrac{2y+1-t}{t+1}\Big), & y \in (-1,t), \\[2ex] \dfrac{-2}{1-t}\, P_{k-1}^{(1,0)}\Big(\dfrac{1+t-2y}{1-t}\Big), & y \in (t,1). \end{cases}$$

Let us underline the identity $\langle s_t, \widehat{s}_t \rangle = 1$ where the normalization $\widehat{s}_t := \dfrac{1-t^2}{8}\, s_t$ is used. Thus, associating $u = \dfrac{2x-1-t}{1-t} \in [-1,1]$ with $x \in [t,1]$, the orthogonal projection of a function f onto $\mathrm{span}\big[s_t\big]$ is

$$\begin{aligned}
P_{\mathrm{span}[s_t]}(f)(x) &= \langle s_t, f\rangle\, \widehat{s}_t(x) \\
&= -\frac{1+t}{4} P_{k-1}^{(1,0)}(-u) \bigg[\int_{-1}^{t} \frac{2}{1+t} P_{k-1}^{(1,0)}\Big(\frac{2y+1-t}{t+1}\Big)\, f(y)\, dy + \int_{t}^{1} \frac{-2}{1-t} P_{k-1}^{(1,0)}\Big(\frac{1+t-2y}{1-t}\Big)\, f(y)\, dy \bigg] \\
&= -\frac{1+t}{4} P_{k-1}^{(1,0)}(-u) \bigg[\int_{-1}^{1} P_{k-1}^{(1,0)}(v)\, f_1(v)\, dv - \int_{-1}^{1} P_{k-1}^{(1,0)}(-v)\, f_2(v)\, dv \bigg],
\end{aligned}$$

where f_1 and f_2 stand for the restrictions of f to $[-1,t]$ and $[t,1]$, transposed to the interval $[-1,1]$. Besides, the orthogonal projection of f onto $\mathcal{S}_{k,0}(\delta)$ is

$$P_{\mathcal{S}_{k,0}(\delta)}(f)(x) = P_{\mathcal{P}_{k,0}}(f_2)(u) = \int_{-1}^{1} \mathcal{K}_{k,0}(u,v)\, f_2(v)\, dv$$

It follows that the orthogonal projection of f onto $\mathcal{S}_{k,1}(\delta)$ is

$$\begin{aligned}
P_{\mathcal{S}_{k,1}(\delta)}(f)(x) &= P_{\mathcal{S}_{k,0}(\delta)}(f)(x) - P_{\mathrm{span}[s_t]}(f)(x) \\
&= \int_{-1}^{1} \frac{1+t}{4} P_{k-1}^{(1,0)}(-u) P_{k-1}^{(1,0)}(v)\, f_1(v)\, dv + \int_{-1}^{1} \Big[\mathcal{K}_{k,0}(u,v) - \frac{1+t}{4} P_{k-1}^{(1,0)}(-u) P_{k-1}^{(1,0)}(-v) \Big]\, f_2(v)\, dv.
\end{aligned}$$

Since f_1 and f_2 can be chosen arbitrarily, we deduce with the help of the symmetry relation for Jacobi polynomials that

$$L_t(x) = \int_{-1}^{1} \Big| \frac{1+t}{4} P_{k-1}^{(0,1)}(u) P_{k-1}^{(0,1)}(v) \Big|\, dv + \int_{-1}^{1} \Big| \mathcal{K}_{k,0}(u,v) - \frac{1+t}{4} P_{k-1}^{(0,1)}(u) P_{k-1}^{(0,1)}(v) \Big|\, dv.$$

The remaining ingredients are the identities $(5)_{m=0}$ and $(4)_{m=0}$, that is

$$\begin{aligned}
\mathcal{K}_{k,0}(u,v) &= \mathcal{K}_{k,1}(u,v) + \frac{1}{2}\, P_{k-1}^{(0,1)}(u)\, P_{k-1}^{(0,1)}(v), \\
\mathcal{K}_{k,0}(1,v) &= \frac{k}{2}\, P_{k-1}^{(1,0)}(v) \;=\; (-1)^{k-1}\, k\, \big[\mathcal{K}_{k,0}(1,-v) - \mathcal{K}_{k,1}(1,-v) \big].
\end{aligned}$$

Hence, we may write

$$L_t(x) = \int_{-1}^{1} \Big| \frac{1+t}{2} \big[\mathcal{K}_{k,0}(u,v) - \mathcal{K}_{k,1}(u,v) \big] \Big|\, dv + \int_{-1}^{1} \Big| \frac{1-t}{2}\, \mathcal{K}_{k,0}(u,v) + \frac{1+t}{2}\, \mathcal{K}_{k,1}(u,v) \Big|\, dv.$$

We could observe that $L_t(t) = \sigma_{k,0}$ independently of t, but we are more interested in

$$\lim_{\tau \to 1} L_\tau(1) = \frac{1}{k} \int_{-1}^{1} \big| \mathcal{K}_{k,0}(1,v) \big|\, dv + \int_{-1}^{1} \big| \mathcal{K}_{k,1}(1,v) \big|\, dv = \frac{\rho_{k,0}}{k} + \rho_{k,1} = \Big[\frac{1}{k}\Big] \sigma_{k,0} + \Big[1 - \frac{1}{k}\Big] \sigma_{k,1},$$

which settles one part of (6). As for the other part, observe that

$$L_t(x) \le \frac{1+t}{2}\Big(\int_{-1}^{1} \big|\mathcal{K}_{k,0}(u,v) - \mathcal{K}_{k,1}(u,v)\big|\, dv + \int_{-1}^{1} \big|\mathcal{K}_{k,1}(u,v)\big|\, dv\Big) + \frac{1-t}{2}\Big(\int_{-1}^{1} \big|\mathcal{K}_{k,0}(u,v)\big|\, dv\Big).$$

Since the second term is known to be maximized at $u = 1$, it is sufficient to establish that the first term is maximized at $u = 1$, for the inequality $\sigma_{k,0} \le \sigma_{k,1}$ would allow to write

$$L_t(x) \le \frac{1+t}{2}\Big(\Big[\frac{1}{k}\Big]\sigma_{k,0} + \Big[1 - \frac{1}{k}\Big]\sigma_{k,1}\Big) + \frac{1-t}{2}\Big(\sigma_{k,0}\Big) \le \Big[\frac{1}{k}\Big]\sigma_{k,0} + \Big[1 - \frac{1}{k}\Big]\sigma_{k,1}.$$

Conversely, because $\lim_{t\to 1} L_t(x) \le \lim_{t\to 1} L_t(1)$ must also hold, we see that it is also necessary to establish that the first term is maximized at $u = 1$. Our equivalence claim is now shown.

Remark. Without any restriction on the number of interior breakpoints but still in the case $m = 1$, Kayumov [5] mentioned — in different notation — that some of his computations support the conjecture that $\Lambda_{k,1} = \sigma_{k,1}$ for $k \le 31$.

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