Some Comments on the Comparison Between Condition Numbers and Projection Constants

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Abstract. The condition number of a space, i.e. the least condition number of its bases, is investigated in connection with minimal and minimal interpolating projections. It is identified as projection constants of particular types. These considerations turn into inequalities between absolute projection constant, Banach–Mazur distance, and absolute interpolating projection constant. Some situations where the inequalities can or cannot become equalities are explored through small-dimensional examples.

§1. Introduction

This paper will give a brief account of the relations between minimal projections of various types and best conditioned bases. The notion of best conditioned basis is less familiar than the one of minimal projection, due to an unusual terminology which is to be clarified at once.

Let V denote in the whole paper a normed space of finite dimension n. It will here and there be considered a subspace of a normed space X, most of the time the space $\mathcal{C}(K)$ of continuous functions on a compact Hausdorff set K. For a basis $\underline{v} = (v_1, \ldots, v_n)$ of V, its ℓ_∞-condition number is defined to be

$$\kappa_\infty(\underline{v}) := \sup_{a \neq 0} \frac{\|\sum_i a_i v_i\|}{\|a\|_\infty} \times \sup_{a \neq 0} \frac{\|a\|_\infty}{\|\sum_i a_i v_i\|}.$$

The first supremum is the maximum of a convex function on the unit ball of ℓ_∞^n, hence it is attained at one of the extreme points of this ball. As for the second supremum, if we denote the basis of V^* dual to $\underline{v}$ by $\underline{\mu} = (\mu_1, \ldots, \mu_n)$, it is simply the maximum of $\max_i |\mu_i(v)|$ over all v in the unit ball of V. The expression for the ℓ_∞-condition number of the

Approximation Theory XII: San Antonio 2007
Marion Neamtu and Larry L. Schumaker (eds.), pp. 1–6.

ISBN 978-0-9728482-9-9

basis $\underline{v}$ therefore simplifies to

$$\kappa_\infty(\underline{v}) = \max_{\varepsilon_i = \pm 1} \Big\| \sum_i \varepsilon_i v_i \Big\| \times \max_i \|\mu_i\|. \tag{1}$$

When computing with the basis $\underline{v}$, we should try to control the error propagation, hence we require $\underline{v}$ to be 'well conditioned'. Naturally enough, we would like to use the 'best conditioned' bases, that is the bases of V whose condition numbers equal the condition number $\kappa_\infty(V)$ of the space V defined by

$$\kappa_\infty(V) := \inf \big\{ \kappa_\infty(\underline{v}), \quad \underline{v} \ \text{ basis of } V \big\}.$$

Best conditioned bases always exist in finite dimension. They are invariant under scaling or isometry, but a best conditioned basis is not necessarily deduced from another one in such a way, see Appendix 5.1. Our notations somehow disobey conventions, for the quantity $\kappa_\infty(V)$ is nothing else than the Banach–Mazur distance from V to ℓ_∞^n, i.e.

$$d(V, \ell_\infty^n) := \inf\{\|T\| \times \|T^{-1}\|, \, T : \ell_\infty^n \to V \text{ isomorphism}\}.$$

In addition to computational aspects, determining the condition numbers of some specific function spaces is also motivated by significant links with minimal projections. This is illustrated by the chain of inequalities

$$p(V) \le \kappa_\infty(V) \le p_{\rm int}(V), \tag{2}$$

involving the projection constant of the space V, its condition number, and its interpolating projection constant.

In Section 2 the definitions of various projection constants will be either recalled or put forward. Then the condition number $\kappa_\infty(V)$ will be identified as some of them in Theorems 1 and 3. A justification of (2) will also be given. In Sections 3 and 4, the inequalities (2) will be examined more closely. Precisely, two problems will be addressed: is it possible for the first inequality to be strict while the second one turns into an equality, and is it possible for the second inequality to be strict while the first one turns into an equality? The former question will be answered affirmatively in Section 3, while the latter will merely be discussed in Section 4, where some related questions are raised.

§2. Basic Results

In this section we present several results needed later in the paper.

2.1. Best normalization

Given a basis $\underline{v} = (v_1, \ldots, v_n)$ of V, a simple choice of the coefficients $\alpha_1, \ldots, \alpha_n$ minimizes the condition number of the basis $(\alpha_1 v_1, \ldots, \alpha_n v_n)$, namely the one that makes the dual functionals all of equal norm, say of norm one. Precisely, if $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ denotes the basis of V^* dual to the basis $\underline{v}$, we introduce the renormalized basis $\underline{v}^N$ defined by $v_i^N := \|\mu_i\| \cdot v_i$, for which the dual basis $\underline{\mu}^N$ satisfies $\mu_i^N = \mu_i / \|\mu_i\|$. It follows, according to (1), that

$$\kappa_\infty(\underline{v}^N) = \max_{\varepsilon_i = \pm 1} \Big\| \sum_i \varepsilon_i \|\mu_i\| v_i \Big\| \le \max_{\varepsilon_i = \pm 1} \Big\| \sum_i \varepsilon_i v_i \Big\| \times \max_i \|\mu_i\| = \kappa_\infty(\underline{v}), \quad (3)$$

i.e., the basis $\underline{v}^N$ is always better conditioned than the basis $\underline{v}$. This elementary observation was already made in [4]. Duality arguments show that a basis is optimally ℓ_1-normalized when its elements are all of equal norm. This is the case, for instance, of the L_1-normalized B-spline basis. In particular, in the space $\mathcal{L}_k$ of polynomials of degree at most k endowed with the L_1-norm on $[-1, 1]$, the usual Bernstein basis $\underline{B} = (B_0, \ldots, B_k)$ is optimally ℓ_1-normalized, and Lyche and Scherer showed in [7] that its ℓ_1-condition number $\kappa_1(\underline{B})$ behaves like $\sqrt{\pi}\, 2^k / \sqrt{k}$.

The observation (3) implies that, when dealing with a best conditioned basis, we can always assume that the elements of the dual basis are all of norm one. This will translate into Theorem 1. It also allows an elegant proof of the known fact that $d(\ell_2^n, \ell_\infty^n) = \sqrt{n}$, together with a characterization of best conditioned bases of ℓ_2^n. It is worth pointing out that no such characterization exists for an arbitrary space V.

Proposition 1. *A basis $\underline{v}$ of the n-dimensional Euclidean space $\mathcal{E}_n$ is best conditioned if and only if it is, up to a scaling factor, orthonormal. Furthermore, the condition number of $\mathcal{E}_n$ is*

$$\kappa_\infty(\mathcal{E}_n) = \sqrt{n}.$$

Proof: Let $\underline{v} = (v_1, \ldots, v_n)$ be a basis of $\mathcal{E}_n$. We are going to prove that $\kappa_\infty(\underline{v}) \ge \sqrt{n}$, with equality if and only if $\underline{v}$ is orthonormal, up to a scaling factor. Let $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ be the basis of $\mathcal{E}_n^*$ dual to $\underline{v}$. Each μ_i can be represented as $\mu_i = \langle u_i, \bullet \rangle$ for some $u_i \in \mathcal{E}_n$. Then, making use of the parallelogram law, we obtain

$$\begin{aligned}\kappa_\infty(\underline{v})^2 \ge \kappa_\infty(\underline{v}^N)^2 &= \max_{\varepsilon_i = \pm 1} \Big\| \sum_i \varepsilon_i \|\mu_i\| v_i \Big\|^2 = \max_{\varepsilon_i = \pm 1} \Big\| \sum_i \varepsilon_i \|u_i\| v_i \Big\|^2 \\ &\ge \frac{1}{2^n} \sum_{\underline{\varepsilon}} \Big\| \sum_i \varepsilon_i \|u_i\| v_i \Big\|^2 = \sum_i \|u_i\|^2 \|v_i\|^2 \ge \sum_i \langle u_i, v_i \rangle^2 = n.\end{aligned}$$

Equalities all the way through imply that $\underline{v}$ is optimally normalized, thus $\|\mu_i\| = \|u_i\| = 1$, and that $\langle u_i, v_i \rangle = \|u_i\|\|v_i\| = 1$, thus $u_i = v_i$, and finally the duality conditions $\langle v_i, v_j \rangle = \delta_{i,j}$ mean that the basis $\underline{v}$ is orthonormal. Conversely, if a basis $\underline{v}$ is orthonormal, it is clear that $\kappa_\infty(\underline{v})^2 = n$. □

2.2. Projection constants

Let us recall that the (relative) projection constant $p(V, X)$ of V in the superspace X – also denoted $\lambda(V, X)$ – is

$$p(V, X) := \inf \big\{ \|P\|,\ P : X \twoheadrightarrow V \text{ projection} \big\},$$

and that the (absolute) projection constant of V is

$$p(V) := \sup \big\{ p(V, X),\ V \text{ isometrically embedded in } X \big\}.$$

Note the common abuse of notation here: $p(V, X)$ stands for $p(i(V), X)$, where $i : V \hookrightarrow X$ is the isometric embedding. With B_{V^*} denoting the unit ball of V^* endowed with the weak*-topology, a typical example is the isometric embedding from V into $\mathcal{C}(B_{V^*})$ induced by the map

$$v \in V \mapsto \big[\, \bullet(v) := \big(\lambda \in V^* \mapsto \lambda(v) \in \mathbb{R} \big) \,\big]. \tag{4}$$

It is important to keep in mind that the equality $p(V) = p(V, \mathcal{C}(K))$ is valid whenever $V \hookrightarrow \mathcal{C}(K)$. Under this hypothesis, we can also define the interpolating projection constant of V in $\mathcal{C}(K)$ by

$$p_{\text{int}}(V, \mathcal{C}(K)) := \inf \big\{ \|P\|,\ P : \mathcal{C}(K) \twoheadrightarrow V \text{ interpolating projection} \big\}.$$

Just as $p(V, \mathcal{C}(B_{V^*}))$ reduces to $p(V)$, we now claim that $p_{\text{int}}(V, \mathcal{C}(B_{V^*}))$ reduces to $\kappa_\infty(V)$. This provides another interpretation of the classical inequality $p(V) \le \kappa_\infty(V)$. The strict inequality will be briefly discussed in Appendix 5.2.

Theorem 1. *The condition number of the space V can be interpreted in terms of projections only, namely one has*

$$\kappa_\infty(V) = p_{\text{int}}(V, \mathcal{C}(B_{V^*})).$$

Proof: We only sketch the general inequality $\kappa_\infty(V) \le p_{\text{int}}(V, \mathcal{C}(K))$, as a stronger version appears in Theorem 3. Let $Pf = \sum_i f(t_i)\ell_i$ represent an interpolating projection at the points $t_1, \ldots, t_n$. Since the norms of dual functionals $v \in V \mapsto v(t_i) \in \mathbb{R}$ are at most 1, the condition number of the Lagrange basis $\underline{\ell} := (\ell_1, \ldots, \ell_n)$ satisfies

$$\kappa_\infty(\underline{\ell}) \le \max_{\varepsilon_i = \pm 1} \Big\| \sum\nolimits_i \varepsilon_i \ell_i \Big\| = \Big\| \sum\nolimits_i |\ell_i| \Big\|.$$

The latter is precisely the norm of $\|P\|$, see e.g. (6). Thus we obtain $\kappa_\infty(V) \le \|P\|$, and it remains to take the infimum over P.

In order to establish the inequality $p_{\rm int}(V, \mathcal{C}(B_{V^*})) \le \kappa_\infty(V)$, let us now consider a best conditioned basis $\underline{v} = (v_1, \dots, v_n)$ of V. We may assume that $\|\mu_i\| = 1$, where $\underline{\mu} = (\mu_1, \dots, \mu_n)$ denotes the basis of V^* dual to $\underline{v}$. Then the expression $Pf = \sum_i f(\mu_i) v_i$ defines an interpolating projection P form $\mathcal{C}(B_{V^*})$ onto V. As such, its norm is given by

$$\|P\| = \Big\| \sum_i |v_i| \Big\| = \max_{\varepsilon_i = \pm 1} \Big\| \sum_i \varepsilon_i v_i \Big\| = \kappa_\infty(\underline{v}).$$

We obtain $p_{\rm int}(V, \mathcal{C}(B_{V^*})) \le \kappa_\infty(\underline{v}) = \kappa_\infty(V)$, as expected. $\square$

Let us consider now the space $\mathcal{P}_2$ of quadratic polynomials endowed with the supremum norm on $[-1, 1]$. According to [4], it is known that $p_{\rm int}(\mathcal{P}_2, \mathcal{C}[-1, 1]) = 5/4$ and that $\kappa_\infty(\mathcal{P}_2) \lessapprox 1.248394563$ (this is likely to be an equality). Theorem 1 therefore implies that the interpolating projection constant $p_{\rm int}(V, \mathcal{C}(K))$ is not independent of $\mathcal{C}(K)$, contrary to the projection constant $p(V, \mathcal{C}(K))$. We suggest to introduce the following interpolating projection constant which is intrinsic to the space V.

Definition 1. *The (absolute) interpolating projection constant of V is*

$$p_{\rm int}(V) := \sup \big\{ p_{\rm int}(V, \mathcal{C}(K)),\ V \hookrightarrow \mathcal{C}(K) \big\}.$$

The inequality $\kappa_\infty(V) \le p_{\rm int}(V)$ mentioned in (2) now comes for free with Theorem 1. The supremum is actually achieved for a specific choice of K, as shown below.

Theorem 2. *Let $E_{V^*} := {\rm cl}[{\rm Ex}(B_{V^*})]$ be the weak*-closure of the extreme points of B_{V^*}. One has*

$$p_{\rm int}(V) = p_{\rm int}(V, \mathcal{C}(E_{V^*})).$$

Proof: Let us first point out that ${\rm Ex}(B_{V^*})$ is a norm-determining set for V, i.e. that for any $v \in V$, there exists $\lambda \in {\rm Ex}(B_{V^*})$ such that $\lambda(v) = \|v\|$. It follows that E_{V^*} is a weak*-compact norm-determining set for V, which justifies that the map (4) induces an isometric embedding from V into $\mathcal{C}(E_{V^*})$. As a matter of fact, the set E_{V^*} is the smallest weak*-compact norm-determining set for V (this fact, not restricted to spaces of finite dimension, often appears implicitly, see e.g. [5]). Thus, if $V \hookrightarrow \mathcal{C}(K)$, we may write $E_{V^*} \subseteq K$. This is again an abuse of notation, which says that any element λ of E_{V^*} is (up to a sign) of the form $\bullet(t)_{|V}$ for some $t \in K$. Now if $Pf = \sum_i f(\lambda_i) \ell_i$ defines an interpolating projection from $\mathcal{C}(E_{V^*})$ onto V, we have $\|P\| = \max_{\varepsilon_i = \pm 1} \| \sum_i \varepsilon_i \ell_i \|$. Since each λ_i can be represented as $\bullet(t_i)_{|V}$ for some $t_i \in K$, we observe that $\widetilde{P} f := \sum f(t_i) \ell_i$

defines an interpolating projection $\widetilde{P}$ from $\mathcal{C}(K)$ onto V. As such, its norm is $\|\widetilde{P}\| = \max_{\varepsilon_i=\pm 1} \| \sum_i \varepsilon_i \ell_i \|$. It follows that $p(V,\mathcal{C}(K)) \le \|\widetilde{P}\| = \|P\|$, and we finally obtain $p(V,\mathcal{C}(K)) \le p(V,\mathcal{C}(E_{V^*}))$ by taking the infimum over P. This is the required result. □

2.3. Connection with generalized interpolating projections

As established in [4], the condition number of the space V is closely related to the generalized interpolating projection constant of V in $\mathcal{C}(K)$, as defined by

$$p_{\text{g.int}}(V,\mathcal{C}(K)) := \inf \big\{ \|P\|, P : \mathcal{C}(K) \twoheadrightarrow V \ \text{g}^{\text{zed}} \text{ interpolating projection} \big\}.$$

The theorem and its proof are reproduced here for completeness, hence a few reminders are necessary. The term generalized interpolating projection was devised in [3] to designate a projection P from $\mathcal{C}(K)$ onto V which can be represented, for some $v_1,\ldots,v_n \in V$ and $\widetilde{\mu}_1,\ldots,\widetilde{\mu}_n \in \mathcal{C}(K)^*$, as

$$Pf = \sum_i \widetilde{\mu}_i(f)\, v_i, \qquad \widetilde{\mu}_i(v_j) = \delta_{i,j}, \qquad \operatorname{car} \widetilde{\mu}_i \text{ pairwise disjoint.} \tag{5}$$

The carrier $\operatorname{car} \widetilde{\mu}$ of a linear functional $\widetilde{\mu}$ on $\mathcal{C}(K)$ is the smallest closed subset C of K such that $f_{|C} = 0 \Rightarrow \widetilde{\mu}(f) = 0$. The simple expression of the norm of a generalized interpolating projection is in fact fundamental, it reads

$$\|P\| = \Big\| \sum_i \|\widetilde{\mu}_i\| \, |v_i| \Big\|_\infty . \tag{6}$$

Theorem 3. *In general, one has the inequality*

$$\kappa_\infty(V) \le p_{\text{g.int}}(V,\mathcal{C}(K)),$$

and if one can choose K to be e.g. $[-1,1]$, then one has the equality

$$\kappa_\infty(V) = p_{\text{g.int}}(V,\mathcal{C}[-1,1]).$$

Proof: Let P be a generalized interpolating projection of the form (5). Note that the system $\underline{\mu} = (\mu_1,\ldots,\mu_n)$, with $\mu_i := \widetilde{\mu}_{i|V}$, constitutes the basis of V^* dual to $\underline{v}$. Since $\|\widetilde{\mu}_i\| \ge \|\mu_i\|$, we get

$$\|P\| \ge \Big\| \sum_i \|\mu_i\| \, |v_i| \Big\| = \kappa_\infty(\underline{v}^N) \ge \kappa_\infty(V).$$

The inequality $\kappa_\infty(V) \le p_{\text{g.int}}(V,\mathcal{C}(K))$ is derived by taking the infimum over P.

Let us now proceed with the reverse inequality in the case $K = [-1,1]$. We consider a basis $(v_1,\ldots,v_n)$ of V. Let $\underline{\mu} = (\mu_1,\ldots,\mu_n)$ denote the basis of V^* dual to $\underline{v}$. Each linear functional μ_i admits a norm-preserving extension to $\mathcal{C}(K)$ of the form (see [9, Theorem 2.13] or [5])

$$\widetilde{\mu}_i(f) = \sum_{j=1}^{n} \alpha_{i,j}\, f(t_{i,j}), \qquad t_{i,j} \in [-1,1]. \tag{7}$$

There exist sequences of points $(t_{i,j}^k)$ converging to $t_{i,j}$ and such that the $t_{i,j}^k$ are all distinct for any fixed k. We consider the linear functional defined on $\mathcal{C}[-1,1]$ by

$$\widetilde{\mu}_i^k(f) := \sum_{j=1}^{n} \alpha_{i,j}\, f(t_{i,j}^k),$$

and we set $\mu_i^k := \widetilde{\mu}_{i|V}^k$. Since $\|\mu_i^k\| \le \|\widetilde{\mu}_i^k\| = \sum_j |\alpha_{i,j}| = \|\widetilde{\mu}_i\| = \|\mu_i\|$, we may assume that (μ_i^k), or at least one of its subsequences, converges in norm. The limit must be μ_i because $\mu_i^k(v) \to \mu_i(v)$ for any $v \in V$. We write $[\mu_1^k, \ldots, \mu_n^k]^\top =: A_k\, [\mu_1, \ldots, \mu_n]^\top$, so that the matrix A_k tends to the identity matrix I. In particular, the matrix A_k is invertible, at least for k large enough, and its inverse A_k^{-1} tends to I. We then define $[v_1^k, \ldots, v_n^k]^\top := A_k^{-1}\, [v_1, \ldots, v_n]^\top$, and we obtain $v_i^k \to v_i$. In view of $\mu_i^k(v_j^k) = \delta_{i,j}$, the expression $P^k f := \sum_i \widetilde{\mu}_i^k(f)\, v_i^k$ defines a generalized interpolating projection from $\mathcal{C}[-1,1]$ onto V. We get

$$\begin{aligned} p_{\mathrm{g.int}}(V, \mathcal{C}[-1,1]) &\le \lim_{k\to\infty} \|P^k\| = \lim_{k\to\infty} \Big\| \sum_i \|\widetilde{\mu}_i^k\|\, |v_i^k| \Big\| \\ &= \lim_{k\to\infty} \Big\| \sum_i \|\mu_i\|\, |v_i^k| \Big\| = \Big\| \sum_i \|\mu_i\|\, |v_i| \Big\| = \kappa_\infty(\underline{v}^N) \le \kappa_\infty(\underline{v}). \end{aligned}$$

The inequality $p_{\mathrm{g.int}}(V, \mathcal{C}[-1,1]) \le \kappa_\infty(V)$ is finally derived by taking the infimum over $\underline{v}$. $\square$

Observe that the space V, being separable, can always be isometrically embedded into $\mathcal{C}[-1,1]$. This implies in particular that the P_λ-problem, asking for a relationship between the projection constant $p(V)$ and the Banach–Mazur distance $d(V, \ell_\infty^n)$, may very well be reformulated in terms of projections only – a remark that could have been made after Theorem 1. The observation also offers a straightforward way of explaining the chain of inequalities (2), which is to be the center of our attention from now on. Precisely, we address two problems:

- Problem 1: Is it possible to have $p(V) < \kappa_\infty(V) = p_{\mathrm{int}}(V)$?
- Problem 2: Is it possible to have $p(V) = \kappa_\infty(V) < p_{\mathrm{int}}(V)$?

§3. Regarding the First Problem

In this section we will present several examples showing that the first question has an affirmative answer. Of crucial importance is the following lemma.

Lemma 1. *If the space V is smooth, then*

$$\kappa_\infty(V) = p_{\text{int}}(V).$$

Proof: The smoothness of V implies the strict convexity of V^*, hence the fact that $B_{V^*} = E_{V^*}$. The result is now an immediate consequence of Theorems 1 and 2. □

Let us illustrate this point with more concrete examples. The subspace of $\mathcal{C}(\mathbb{T})$ spanned by cos and sin is the simplest one. Let us denote it $\mathcal{E}_2$, since it is Euclidean by virtue of

$$\|a\cos + b\sin\|_\infty = \sqrt{a^2+b^2}.$$

Thus we know that $\kappa_\infty(\mathcal{E}_2) = p_{\text{int}}(\mathcal{E}_2) = \sqrt{2}$. On the other hand, we know that $p(\mathcal{E}_2) = 4/\pi$, as a particular case of

$$p(\mathcal{E}_n) = \frac{2}{\sqrt{\pi}} \frac{\Gamma\big((n+2)/2\big)}{\Gamma\big((n+1)/2\big)}.$$

Alternatively, $p(\mathcal{E}_2) = 4/\pi$ can be deduced from the minimality of the orthogonal projection onto $\mathcal{E}_2$ – this is justified exactly in the same way as the minimality of the Fourier projection onto the space $\mathcal{T}_n$ of trigonometric polynomials of degree at most n. In any case, we emphasize that

$$p(\mathcal{E}_2) < \kappa_\infty(\mathcal{E}_2) = p_{\text{int}}(\mathcal{E}_2).$$

Let us consider next the space $\mathcal{L}_n$ of algebraic polynomials of degree at most n endowed with the L_1-norm on $[-1,1]$. Let us specify $n=1$ and recount some history about the determination of $p(\mathcal{L}_1)$. First, Franchetti and Cheney obtained the value $p(\mathcal{L}_1, L_1) \approx 1.22040491$ in [6]. Chalmers then remarked in [2] that this relative projection constant actually equals the absolute projection constant. To this end, he noticed that

$$\int_{-1}^{1} |at+b|dt = \max_{t\in[-1,1]} |a(1-t^2)+2bt| = \begin{cases} 2|b|, & \text{if } |a| \le |b|, \\ \dfrac{a^2+b^2}{|a|}, & \text{if } |a| > |b|, \end{cases}$$

so that the space $\mathcal{L}_1$ spanned by $v_1(t) := t$ and $v_2(t) := 1/2$ is isometrically isomorphic to the space $\mathcal{W}$ spanned by $w_1(t) := 1-t^2$ and $w_2(t) := t$, and endowed with the supremum norm on $[-1, 1]$. He could conclude that $p(\mathcal{L}_1, L_1) = p(\mathcal{L}_1)$ by outlining a projection from $\mathcal{C}[-1, 1]$ onto $\mathcal{W}$ with norm $p(\mathcal{L}_1, L_1)$, so that

$$p(\mathcal{L}_1) = p(\mathcal{W}) = p(\mathcal{W}, \mathcal{C}[-1, 1]) \leq p(\mathcal{L}_1, L_1) \leq p(\mathcal{L}_1).$$

We shall now remark that the space $\mathcal{L}_n$ is smooth, since the measure of the set $\{x \in [-1, 1] : v(x) = 0\}$ equals zero for any $v \in V \setminus \{0\}$. This yields $\kappa_\infty(\mathcal{L}_n) = p_{\mathrm{int}}(\mathcal{L}_n)$. The precise value is $\kappa_\infty(\mathcal{L}_1) = p_{\mathrm{int}}(\mathcal{L}_1) = 5/4$ for $n = 1$. This is obtained by minimizing the norm of an interpolating projection from $\mathcal{C}[-1, 1]$ onto $\mathcal{W}$. Note that $\kappa_\infty(\mathcal{L}_1) = 5/4$ was obtained in [5] without the help of Lemma 1, at the price of six pages of intricate calculations. In any case, we emphasize that

$$p(\mathcal{L}_1) < \kappa_\infty(\mathcal{L}_1) = p_{\mathrm{int}}(\mathcal{L}_1).$$

Our final example indicates that the condition of smoothness of V is not necessary for the equality $\kappa_\infty(V) = p_{\mathrm{int}}(V)$ to hold. It will be provided by the subspace $\mathcal{T}_1$ of $\mathcal{C}(\mathbb{T})$, where $\mathcal{T}_n$ denotes the space of all trigonometric polynomials of degree at most n. Clearly, this space is not smooth, for a constant function has many support functionals. As shown in [1], the Lagrange bases at equidistant points are associated with minimal interpolating projections, and in the case $n = 1$ computations even reveal that these bases are best conditioned (no such computations have been carried out for $n \geq 2$). We infer that $\kappa_\infty(\mathcal{T}_1) = p_{\mathrm{int}}(\mathcal{T}_1, \mathcal{C}(\mathbb{T})) = 5/3$. We also have to make sure that $E_{\mathcal{T}_1^*} = \mathbb{T}$, which presents no difficulty. On the other hand, the minimality of the Fourier projection onto $\mathcal{T}_n$ implies that $p(\mathcal{T}_1, \mathcal{C}(\mathbb{T})) = 1/3 + 2\sqrt{3}/\pi \approx 1.435991124$. To conclude, we emphasize once more that

$$p(\mathcal{T}_1) < \kappa_\infty(\mathcal{T}_1) = p_{\mathrm{int}}(\mathcal{T}_1).$$

§4. Regarding the Second Problem

The second problem will not be settled in this section, only discussed. We have never come across the situation $p(V) = \kappa_\infty(V) < p_{\mathrm{int}}(V)$, and it is our belief that the equality $p(V) = \kappa_\infty(V)$ should imply the equality $p(V) = p_{\mathrm{int}}(V)$. Note that $p(V) > 1$ can be assumed, since $p(V) = 1$ is known to imply $V \cong \ell_\infty^n$, hence $p_{\mathrm{int}}(V) = 1$. At this point, we should at least verify that the hypothesis $p(V) = \kappa_\infty(V) > 1$ is not void. The example provided by Pan and Shektman in [8] does the trick. Indeed, they examined a subspace V of a discrete $\mathcal{C}(K)$, namely $\mathcal{C}(K) = \ell_\infty^6$, satisfying $p(V, \mathcal{C}(K)) = p_{\mathrm{int}}(V, \mathcal{C}(K)) > 1$. This immediately yields $p(V) = \kappa_\infty(V) > 1$. If our belief is true, this should imply that

$p(V) = p_{\text{int}}(V)$, which is not a straightforward consequence of the above. Let us recall that the space V is spanned by vectors v and w defined by

$$\begin{aligned} v &:= [1, 0, \tau, \upsilon, \tau, \upsilon]^\top, \\ w &:= [0, 1, \upsilon, \tau, -\upsilon, -\tau]^\top, \end{aligned}$$

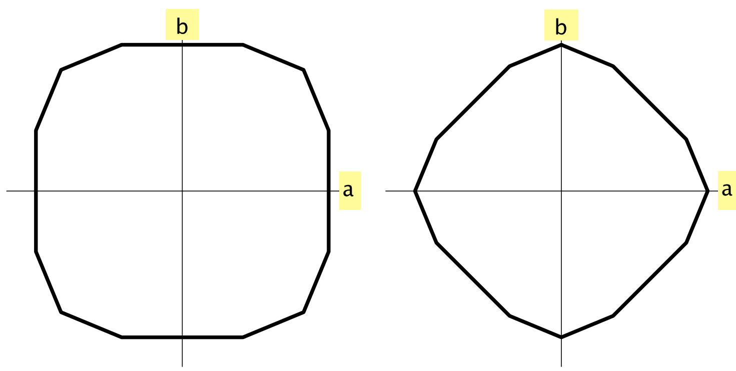
where $\tau := (2+\sqrt{2})/4$ and $\upsilon := \sqrt{2}/4$. We also recall the expression of the projection P from $\mathcal{C}(K) = \ell_\infty^6$ onto V which is both minimal and interpolating, namely

$$P(x) = x_1 v + x_2 w, \qquad \text{for which } \|P\| = \||v| + |w|\| = \frac{1+\sqrt{2}}{2}.$$

We aim at proving that the linear functionals $\lambda : x \in \ell_\infty^6 \mapsto x_1 \in \mathbb{R}$ and $\mu : x \in \ell_\infty^6 \mapsto x_2 \in \mathbb{R}$ are extreme points of $\text{Ex}(B_{V^*})$, so that P also defines an interpolating projection from $\mathcal{C}(E_{V^*})$ onto V. For this purpose, with $\sigma := \sqrt{2} - 1$, we observe that

$$\begin{aligned} \|av + bw\| &= \max[|a|,\ |b|,\ \tau|a| + \upsilon|b|,\ \upsilon|a| + \tau|b|], \\ \|a\lambda + b\mu\| &= \max[|a| + \sigma|b|,\ \sigma|a| + |b|,\ 2\sigma(|a| + |b|)]. \end{aligned}$$

We have used the fact that $\|a\lambda + b\mu\| = \max_{u \in \text{Ex}(B_V)} \|a\lambda(u) + b\mu(v)\|$ in the latter calculation. We can now draw the unit balls of V and V^*, respectively. The figure shows that indeed λ and μ are extreme points of the unit ball of V^*.



To conclude this discussion, we note that we can reformulate the problem $p(V) = \kappa_\infty(V) \overset{?}{\Rightarrow} p(V) = p_{\text{int}}(V)$ in terms of projections only, according to Theorems 1 and 3. Thus, two related questions are worth inquiries, namely

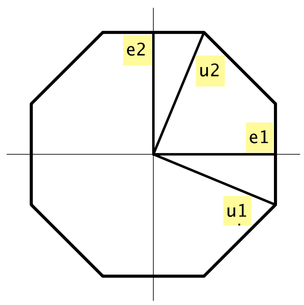
- if a minimal projection from $\mathcal{C}(B_{V^*})$ onto V is interpolating, can we find a minimal projection from $\mathcal{C}(E_{V^*})$ onto V which is interpolating?
- if a minimal projection from $\mathcal{C}[-1,1]$ onto V is generalized interpolating, can we find a minimal projection from $\mathcal{C}[-1,1]$ onto V which is interpolating?

§5. Appendix

5.1. Best conditioned bases are not essentially unique

As announced in Section 1, we shall prove here that not every best conditioned basis is of the form $c\,T(\underline{v})$, where c is a non-zero constant, T is an isometry from V to V, and $\underline{v}$ is a fixed best conditioned basis. The counterexample is supplied by the regular octahedral space, i.e. the space V is taken to be $\mathbb{R}^2$ endowed with the norm

$$\|(a,b)\| := \max[|a|,\ |b|,\ (|a|+|b|)/\sqrt{2}].$$



We claim that $\kappa_\infty(V) = \sqrt{2}$, and that the bases $\underline{e} = (e_1, e_2)$ and $\underline{u} = (u_1, u_2)$, shown on the picture of the unit ball B_V of V, are best conditioned. We then assert that the linear map transforming $\underline{e}$ into $\underline{u}$ is not an isometry of V, since u_1 and u_2 are extreme points of B_V, while e_1 and e_2 are not.

Let $\underline{v} = (v_1, v_2)$ be an arbitrary basis of V, and let $\underline{\mu} = (\mu_1, \mu_2)$ be the basis of V^* dual to $\underline{v}$. We need to establish that $\kappa_\infty(\underline{v}) \geq \sqrt{2}$. Let us denote by $\underline{\lambda} = (\lambda_1, \lambda_2)$ the basis of V^* dual to the canonical basis $\underline{e}$ of V. The bases $\underline{v}$ and $\underline{\mu}$ may be represented as

$$\begin{aligned} v_1 &= \alpha\, e_1 + \beta\, e_2, & \mu_1 &= [\ \ \delta\,\lambda_1 - \gamma\,\lambda_2\,]\,/\,D,\\ v_2 &= \gamma\, e_1 + \delta\, e_2, & \mu_2 &= [-\beta\,\lambda_1 + \alpha\,\lambda_2]\,/\,D, \end{aligned}$$

where $D := \alpha\delta - \beta\gamma \neq 0$. Using $\|\mu_i\| = \max_{u \in \mathrm{Ex}(B_V)} |\mu_i(u)|$, we obtain

$$\begin{aligned} \|\mu_1\| &= \max[|\gamma| + (\sqrt{2}-1)|\delta|,\ (\sqrt{2}-1)|\gamma| + |\delta|]\,/\,D,\\ \|\mu_2\| &= \max[|\alpha| + (\sqrt{2}-1)|\beta|,\ (\sqrt{2}-1)|\alpha| + |\beta|]\,/\,D. \end{aligned}$$

Let us also notice that, with $\varepsilon = \pm 1$,

$$\|v_1 + \varepsilon v_2\| = \max[|\alpha + \varepsilon\beta|,\ |\gamma + \varepsilon\delta|,\ (|\alpha + \varepsilon\beta| + |\gamma + \varepsilon\delta|)/\sqrt{2}].$$

The values $\alpha = 1$, $\beta = 1 - \sqrt{2}$, $\gamma = \sqrt{2} - 1$, $\delta = 1$ correspond to the basis $\underline{u}$. In this case, we can verify that $\|\mu_1\| = \|\mu_2\| = 1$ and that $\|v_1 \pm v_2\| = \sqrt{2}$. This yields $\kappa_\infty(\underline{u}) = \sqrt{2}$, as announced. Besides, we easily check that $\|\lambda_1\| = \|\lambda_2\| = 1$ and that $\|e_1 \pm e_2\| = \sqrt{2}$, so that $\kappa_\infty(\underline{e}) = \sqrt{2}$ holds as well. Back to the arbitrary basis $\underline{v}$, we remark that the expressions of $\|\mu_1\|$, $\|\mu_2\|$, and $\|v_1 + \varepsilon v_2\|$ are invariant under the changes $\alpha \leftrightarrow \beta$ and

$\gamma \leftrightarrow \delta$, so we may safely assume that $|\alpha| \geq |\beta|$ and that $|\gamma| \geq |\delta|$. Since we may also assume that the basis $\underline{v}$ is optimally normalized, the conditions $\|\mu_1\| = \|\mu_2\| = 1$ read

$$|\gamma| + (\sqrt{2}-1)|\delta| \;=\; |\alpha| + (\sqrt{2}-1)|\beta| \;=\; |\alpha\delta - \beta\delta|. \tag{8}$$

We now separate two cases. In each of them, we will get $\kappa_\infty(\underline{v}) \geq \sqrt{2}$, which will complete the proof.

Case 1: $\alpha\beta\gamma\delta > 0$. The fact that $\mathrm{sgn}(\alpha\beta) = \mathrm{sgn}(\gamma\delta)$ implies

$$\max_{\varepsilon=\pm 1} \|v_1 + \varepsilon v_2\| = \max[|\alpha| + |\beta|,\ |\gamma| + |\delta|,\ (|\alpha| + |\beta| + |\gamma| + |\delta|)/\sqrt{2}].$$

Using (8) with e.g. $|\alpha\delta| \geq |\beta\gamma|$, we then get

$$\begin{array}{lcl} |\gamma| & \leq & |\gamma| + (\sqrt{2}-1)|\delta| \\ |\alpha| & \leq & |\alpha| + (\sqrt{2}-1)|\beta| \end{array} \leq |\alpha\delta| \leq |\alpha||\gamma|.$$

The inequalities $|\alpha| \geq 1$ and $|\gamma| \geq 1$ follow. We finally derive that

$$\kappa_\infty(\underline{v}) \geq \frac{|\alpha| + |\beta| + |\gamma| + |\delta|}{\sqrt{2}} \geq \sqrt{2}.$$

Case 2: $\alpha\beta\gamma\delta \leq 0$. Since $\mathrm{sgn}(\alpha\delta) \neq \mathrm{sgn}(\beta\gamma)$, we rewrite (8) as

$$|\gamma| + (\sqrt{2}-1)|\delta| \;=\; |\alpha| + (\sqrt{2}-1)|\beta| \;=\; |\alpha\delta| + |\beta\gamma|.$$

Note that one of $|\alpha|$ or $|\gamma|$ is positive. We suppose e.g. that $|\gamma| > 0$, and we set $x := |\delta|/|\gamma| \in [0,1]$. We have

$$1 + (\sqrt{2}-1)x = |\alpha|x + |\beta|.$$

The linear function $f(x) := (|\alpha| + 1 - \sqrt{2})x + |\beta| - 1$ vanishes on $[0,1]$, so we must have $f(0)f(1) \leq 0$, i.e. $(|\beta| - 1)(|\alpha| + |\beta| - \sqrt{2}) \leq 0$. Thus, if $|\alpha| + |\beta| \leq \sqrt{2}$, we obtain $|\beta| \geq 1$, and as a result $|\alpha| + |\beta| \geq 2|\beta| \geq 2$, a contradiction. This means that $|\alpha| + |\beta| \geq \sqrt{2}$ holds. We finally derive that

$$\kappa_\infty(\underline{v}) \geq |\alpha| + |\beta| \geq \sqrt{2}.$$

5.2. Projection constant strictly smaller than condition number

In connection with the second problem, we should have a look at the restrictions imposed by the equality $p(V) = \kappa_\infty(V)$. Hence we suppose that $p(V,X) = \kappa_\infty(V)$, where the superspace X is not necessarily of the type $\mathcal{C}(K)$. Let $\underline{v} = (v_1, \ldots, v_n)$ be a best conditioned basis of V. We may assume that $\|\mu_i\| = 1$, where $\underline{\mu} = (\mu_1, \ldots, \mu_n)$ is the basis of V^* dual to $\underline{v}$. For each μ_i, we pick a norm-preserving extension $\widetilde{\mu}_i$ to the whole

X. We then define a projection P from X onto V by $Px := \sum_i \widetilde{\mu}_i(x)\, v_i$. We take a slight detour to establish that P is minimal, as follows. For $\lambda \in \mathrm{Ex}(B_{V^*})$ satisfying $\|P^*(\lambda)\| = \|P\|$, we write

$$\begin{aligned}\|P\| = \|P^*(\lambda)\| = \Big\| \sum_i \lambda(v_i)\widetilde{\mu}_i \Big\| \le \sum_i |\lambda(v_i)| =: \sum_i \varepsilon_i \lambda(v_i) \\ = \lambda\Big(\sum_i \varepsilon_i v_i \Big) \le \Big\| \sum_i \varepsilon_i v_i \Big\| \le \kappa_\infty(\underline{v}) = \kappa_\infty(V) = p(V,X) \le \|P\|. \quad (9)\end{aligned}$$

Now, if $X = \mathcal{C}(K)$, let us recall that $\widetilde{\mu}_i$ can be chosen in the form (7). Thus, in addition to being minimal, the projection P is also discrete, in the sense that the carriers of the $\widetilde{\mu}_i$ are finite. We could try to find some conditions under which minimal projections fail to attain their norms, for the norm of a discrete projection is always attained. Coming back to (9), we remark that equalities all the way through imply in particular that

$$\Big\| \sum_i \alpha_i \, \widetilde{\mu}_i \Big\| = 1, \qquad \alpha_i := \frac{\lambda(v_i)}{\sum_j |\lambda(v_j)|}.$$

Assume now that X^* is strictly convex, i.e. that no point of the unit sphere of X^* is a strict convex combination of other points of this unit sphere. We may take for example $X = L_p[-1,1]$ for $1 < p < \infty$. Provided that $n \ge 2$, we must have e.g. $\alpha_1 = \pm 1$, $\alpha_2 = 0, \ldots, \alpha_n = 0$. The linear functional λ is proportional to the linear functional μ_1, and the values of their norms yield $\lambda = \pm \mu_1$. It follows that $\kappa_\infty(V) = \sum_i |\mu_1(v_i)| = 1$, meaning that V is isometrically isomorphic to ℓ_∞^n. Since X^* is strictly convex, however, the space X is smooth and consequently so is V. This is contradictory. We can therefore state the following result, and wonder if the inequality $p(V,X) < p(V)$ would hold as well under the same hypothesis.

Proposition 2. *Let X be a normed space whose dual is strictly convex, and let V be a subspace of X of finite dimension ≥ 2. One has*

$$p(V,X) < \kappa_\infty(V).$$

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