Math 414
Mar. 25, 2020

Last time: Went over basic theorems for LTI filters & discussed causal filters & gave preview of wavelets. [see Sections 2.3.1, 2.3.2 & start of Chapter 4].

Today: Examples of LTI filters & causal filters, IF time, discuss Parseval's thm. and the sampling theorem.

1. LTI examples.

Thm. L is LTI if and only if

$$L[f] = f * h$$

where h is a function that is in $L^1(-\infty,\infty)$ or some other space.

Impulse response — $\delta(t)$ = impulse at $t=0$.

$$L[\delta(t)] = \delta * h = \int_{-\infty}^{\infty} \delta(\tau)\, h(t-\tau)\, d\tau$$

$$" \quad = h(t-0) = h(t)$$

Frequency response (This is proved in the book)

$$L[e^{i\lambda t}] = \sqrt{2\pi}\,\hat{h}(\lambda)\, e^{i\lambda t} \quad \left\{ \begin{array}{l} \sqrt{2\pi}\,\hat{h}(\lambda) \\ = \text{Freq. response.} \end{array} \right.$$

Example 1.

$0 \le t \le 1$

Let $h(t) = \begin{cases} 1, & 0 \le t \le \cancel{\text{some const.}} \\ 0, & \text{otherwise.} \end{cases}$

(a) Find the ~~impulse~~ frequency response funct.

$$\sqrt{2\pi}\,\hat{h}(\lambda) = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t) e^{-i\lambda t}\, dt$$

$$= \int_0^1 e^{-i\lambda t}\, dt$$

$$= \frac{e^{-i\lambda} - e^0}{-i\lambda} = \frac{e^{-i\lambda} - 1}{-i\lambda}$$

$$\sqrt{2\pi}\,\hat{h}(\lambda) = \frac{1 - e^{-i\lambda}}{i\lambda}$$ || Frequency Response.

(b) The time domain

$$L[f](t) = \int_{-\infty}^{\infty} f(t-\tau) h(\tau)\, d\tau = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau.$$

SAME — Choose this.

$$L[f](t) = \int_{-\infty}^{\infty} f(\tau) \begin{cases} 1, & \text{if } 0 \le t-\tau \le 1 \\ 0, & \text{otherwise} \end{cases} d\tau$$

$0 \le t - \tau \le 1 \Rightarrow \tau \le t$ and $\tau \ge t-1$ || $L[f] = \int_{t-1}^{t} f(\tau)\, d\tau$

(c) The frequency domain

$$\widehat{L[f]}(\lambda) = \sqrt{2\pi}\, \hat{f}(\lambda)\, \hat{h}(\lambda)$$

Since (by (a)), $\sqrt{2\pi}\, \hat{h}(\lambda) = \dfrac{1-e^{-i\lambda}}{i\lambda}$,

$$\widehat{L[f]}(\lambda) = \left(\frac{1-e^{-i\lambda}}{i\lambda}\right) \hat{f}(\lambda).$$

Because $\dfrac{1-e^{-i\lambda}}{i\lambda} \to 0$ as $\lambda \to \pm\infty$,

the high frequencies will be damped.

For $|\lambda| \ll 1$,

$$\widehat{L[f]}(\lambda) = \left(\frac{1-\overbrace{\sum_{k=0}^{\infty} \frac{(-i\lambda)^k}{k!}}^{e^{-i\lambda}}}{i\lambda}\right)\hat{f}(\lambda)$$

$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, works for any complex #.

$$= \left(\frac{1-\left(1+\frac{(-i\lambda)}{1!}+\frac{(-i\lambda)^2}{2!}+\cdots\right)}{i\lambda}\right)\hat{f}(\lambda)$$

$$= \hat{f}(\lambda)\left(\frac{i\lambda + \frac{\lambda^2}{2} - \frac{i\lambda^3}{3!}+\cdots}{i\lambda}\right) = \Big(1 \underbrace{- \frac{i\lambda}{2} + \frac{\lambda^2}{6} \cdots}\Big)\hat{f}(\lambda)$$

For $|\lambda| \ll 1$, we may neglect these terms.

$$\approx \hat{f}(\lambda)$$

Thus the low frequencies are passed through.

Low pass filter

Example 2. Let $h(t) = e^{-|t|}$.

(a) Find the frequency response funct.

$$\sqrt{2\pi}\,\hat{h}(\lambda) = \sqrt{2\pi}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|t|}e^{-i\lambda t}\,dt$$

$$\sqrt{2\pi}\,\hat{h}(\lambda) = \int_{-\infty}^{0} e^{-(-t)-i\lambda t}\,dt + \int_{0}^{\infty} e^{-t-i\lambda t}\,dt$$

↑ Split integral

$$\sqrt{2\pi}\,\hat{h}(\lambda) = \int_{-\infty}^{0} e^{(1-i\lambda)t}\,dt + \int_{0}^{\infty} e^{-(1+i\lambda)t}\,dt$$

$$= \left.\frac{e^{(1-i\lambda)t}}{1-i\lambda}\right|_{-\infty}^{0} \cancel{+} \left.\frac{e^{-(1+i\lambda)t}}{(1+i\lambda)}\right|_{0}^{\infty}.$$

Comments: For $t \to \infty$, $\cancel{e^{-(1+i\lambda)t}}$

$$\left|e^{-(1+i\lambda)t}\right| = e^{-t}\cdot\underbrace{\left|e^{-i\lambda t}\right|}_{1}.$$

$$\Rightarrow \lim_{t\to+\infty} e^{-(1+i\lambda)t} = 0$$

Also, for $t < 0$,

$$\left|e^{(1-i\lambda)t}\right| = e^{t}\cdot\underbrace{\left|e^{-i\lambda t}\right|}_{1} = e^{t}$$

~~As~~ ~~$t \to \infty$~~ $t \to -\infty$, $e^{t} \to 0$.

$$\Rightarrow \sqrt{2\pi}\,\hat{h}(\lambda) = \frac{1-0}{1-i\lambda} - \frac{(0-1)}{1+i\lambda} = \frac{1}{1-i\lambda} + \frac{1}{1+i\lambda}$$

$$\Rightarrow \sqrt{2\pi}\,\hat{h}(\lambda) = \frac{1+i\lambda+1-i\lambda}{(1-i\lambda)(1+i\lambda)} = \frac{2}{1+\lambda^2}.$$

(b) Find the time domain filter ~~of~~ of

$$f(t) = \begin{cases} e^{-t}, & t>0 \\ 0, & t<0. \end{cases}$$

$$L[f] = \int_{-\infty}^{\infty} f(\tau)\, e^{-|t-\tau|}\, d\tau = \int_0^{\infty} e^{-\tau-|t-\tau|}\, d\tau.$$

If $t \geq 0$, $$L[f] = \int_0^t e^{-\tau-(t-\tau)}\, d\tau + \int_t^{\infty} e^{-\tau-\tau+t}\, d\tau$$

$$= e^{-t}\int_0^t d\tau + e^{t}\int_t^{\infty} e^{-2\tau}\, d\tau$$

$$= t e^{-t} + e^{t}\left[\cancel{\int_t} \frac{e^{-2\tau}}{2}\right]$$

$$= e^{-t}\left[t + \tfrac{1}{2}\right).$$

If $t \leq 0$, $$\cancel{L[f] = \int_0^t e^{-\tau+t-\tau}\, d\tau + \int_t^{\infty}}$$

$$L[f] = \int_0^{\infty} e^{-\tau+(t-\tau)}\, d\tau + \cancel{\int_0^{\infty} \ldots}$$

$$L[f] = e^{t}\int_0^{\infty} e^{-2\tau}\, d\tau = \tfrac{1}{2} e^{t}.$$

Example 3. The Butterworth filter:

$$\begin{cases} h(t) = H(t)e^{-\alpha t}, \\ H(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \end{cases}$$

(a) Time domain —

$$L[f] = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

since $h(t) = 0$ if $t < 0$, we have $h(t-\tau) = 0$ if $t < \tau$. Thus, ~~the~~ the integral is

$$L[f] = \int_{-\infty}^{t} f(\tau)\, e^{-\alpha(t-\tau)}\, d\tau = e^{-\alpha t}\int_{-\infty}^{t} f(\tau)\, e^{\alpha\tau} d\tau.$$

$$L[f] = e^{-\alpha t}\int_{-\infty}^{t} f(\tau)\, e^{\alpha\tau} d\tau.$$

Note: If $f(t) = 0$ for all $t < 0$, then

$$L[f](t) = 0.$$

(b) Frequency domain $\sqrt{2\pi}\,\hat{h}(\lambda) = \int_0^\infty e^{-\alpha t - i\lambda t}\, dt$

$$\Rightarrow \sqrt{2\pi}\,\hat{h}(\lambda) = \int_0^\infty e^{-(\alpha + i\lambda)t}\, dt = \frac{1}{\alpha + i\lambda} \leftarrow \text{Low pass.}$$

Note: $\sqrt{2\pi}\,\hat{h}(\lambda) =$ ~~Laplace transf~~ Laplace transform of h with $s = i\lambda$.

Example 4

1. Consider the frequency response function

$$\sqrt{2\pi}, \hat{h}(\lambda) = \begin{cases} 1 & -1 \le \lambda \le 1 \\ \text{otherwise} & \end{cases}.$$

(a) Find the impulse response function.

Solution

$$\begin{aligned} h(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(\lambda) e^{\lambda t} d\lambda \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{e^{\lambda t}}{\sqrt{2\pi}} d\lambda \\ &= \frac{1}{2\pi} \frac{e^{it} - e^{-it}}{it} = \frac{\sin(t)}{\pi t}. \end{aligned}$$

(b) Find the frequency domain form of L for any f.

Solution $\widehat{L[f]} = \sqrt{2\pi}\, \hat{h} \hat{f}$, which gives us

$$\widehat{L[f]} = \begin{cases} \hat{f}(\lambda) & |\lambda| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is a low pass filter; it passes all frequencies between $-1 \le \lambda \le 1$ and removes all higher frequencies from f.

(c) Suppose that $f(t) = e^{-|t|}$. Find $L[f]$ in both the time domain and frequency domain.

Solution In a homework problem you showed that $\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\lambda^2}$, so in the frequency domain $\widehat{L[e^{-|t|}]}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\lambda^2}$, for $|\lambda \le 1$, and 0 otherwise. In the time-domain, we have $L[e^{-|t|}] = \int_{-\infty}^{\infty} e^{-|t-u|} \frac{\sin(u)}{\pi u} du$.

(d) Suppose that $f(t) = 1$ for $0 \le t \le 1$ and 0 otherwise. (This is the Haar scaling function.) Find $L[f]$ in both the time domain and frequency domain.

Solution For the frequency domain, we need to find $\hat{f}$:

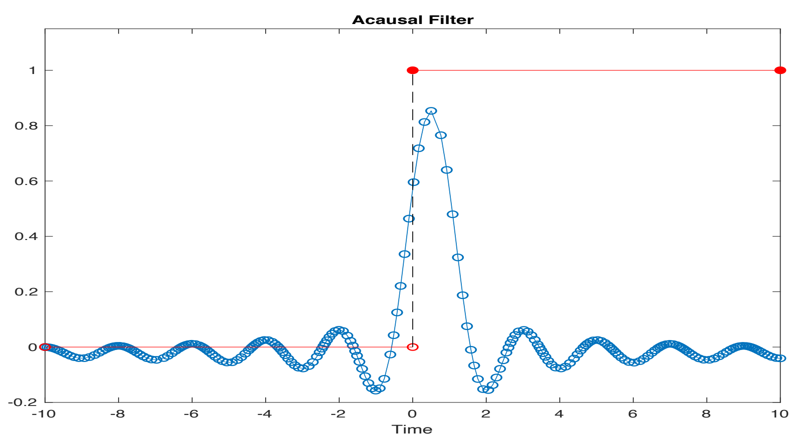
$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-i\lambda t} dt = \frac{1}{\sqrt{2\pi}} \frac{e^{-\lambda} - e^0}{-i\lambda} = \frac{1}{\sqrt{2\pi}} \frac{1 - e^{-i\lambda}}{i\lambda}.$$

Thus, $\widehat{L[f]}(\lambda) = \frac{1}{\sqrt{2\pi}} \begin{cases} \frac{1-e^{-i\lambda}}{i\lambda} & |\lambda| \le 1 \\ 0 & \text{otherwise} \end{cases}$

The situation in the time domain is more interesting:

$$\begin{aligned} L[f] &= \int_{-\infty}^{\infty} f(u) \frac{\sin(t-u)}{\pi(t-u)} du \\ &= \int_0^1 \frac{\sin(t-u)}{\pi(t-u)} du. = \int_{t-1}^{t} \frac{\sin(v)}{\pi v} dv \end{aligned}$$

As the figure below shows, this filter has a big problem. The output is coming in before the input signal does! This violates the fundamental principle of causality: the signal is filtered *before* it arrives!



This violates causality. It implies that the filter "sees" the signal before it arrives. This is science fiction! Such a filter is said to be *acausal*. It can't be realized physically.

Causal filters are ones that don't output a signal before it arrives. If they had a slogan it would be"no output before input." The text gives two equivalent necessary and sufficient conditions for a filter to be causal. The first is that $h(t) = 0$ for all $t < 0$, and the second is that frequency response $\sqrt{2\pi}\hat{h}(\lambda)$ is the Laplace transform of a function (ultimately the impulse response function) evaluated at $s = i\lambda$. For example, the Butterworth filter has its impulse response, $h(t) = 0$, for $t < 0$. Its frequency response is

$$\sqrt{2\pi}\hat{h}(s) = \int_0^\infty e^{-\alpha t}e^{-st}dt, \ s = \lambda.$$

Example 1 above is also causal. Example 2 is acausal.