

Math 414
Apr. 17, 2020

(1)

Last time: Processing a signal and a discussion of the sampling theorem.

Today: Mallat's MRA.

1. Multiresolution analysis. Based on various examples, Stephan Mallat came up with the ~~following~~ following general framework for analyzing a signal at various resolutions.

Defn. Mallat's Multiresolution Analysis (MRA)

Defⁿ. A multiresolution analysis is a collection of subspaces $\{V_j\}_{j=-\infty}^{\infty}$ of $L^2(\mathbb{R})$, together with a scaling function ϕ , if the following properties hold:

1. (Nested) $V_j \subset V_{j+1}$; all j .

2. (Density) closure of $\bigcup V_j = \overline{\bigcup V_j} = L^2(\mathbb{R})$.

→ This means that for any $f \in L^2(\mathbb{R})$ one may find ϕ such that given any $\epsilon > 0$, $\|f - f_\phi\| < \epsilon$. This is an approximation condition.

3. (Separation). $\bigcap V_j = \{0\}$. This just says that the function $f \in L^2(\mathbb{R})$ such that $f \in V_j$ for j is 0.

(2)

4. (Scaling). $f(x) \in V_j$ iff $f(2^{-j}x) \in V_0$.

5. (Orthonormal basis) $\phi \in V_0$ and $\{\phi(x-k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Terms: The V_j 's are called scaling spaces and $\phi(x)$ is the scaling function.

Terms: The V_j 's are called approximation spaces (or scaling spaces). The function ϕ is called the scaling function.

Examples

• Burr MRA. $V_j = \{f \in L^2 : f \text{ is compact in } 2^{-j} \mathbb{Z} \leq x < 2^{-j}(k+1)\}$.

$$\phi(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

— We have examined this case in detail.

• Shannon MRA. V_j is the set of all band-limited signals whose Fourier transforms are zero outside of the interval $[-2^j\pi, 2^j\pi]$. That is, $f \in V_j$ iff $\hat{f}(w) = 0$ for $w \notin [-2^j\pi, 2^j\pi]$,

$$\text{so } f(x) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \hat{f}(w) e^{ixw} dw.$$

The scaling function $\psi(x) = \begin{cases} 1, & x=0 \\ \frac{\sin(x)}{\pi x}, & x \neq 0 \end{cases}$

(3)

Let's look at the properties: (Shannon MRA)

1. Nested. Suppose $f \in V_j$. Then, ~~then~~

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-2^j\pi}^{2^j\pi} f(a) e^{ixa} da. \quad \text{If we let } f(a) = 0$$

on $(-2^{j+1}\pi, 2^{j+1}\pi)$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-2^j\pi}^{2^j\pi} f(a) e^{ixa} da = \frac{1}{\sqrt{2\pi}} \int_{-2^j\pi}^{2^{j+1}\pi} f(a) e^{ixa} da$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2^j\pi}^{2^{j+1}\pi} f(a) e^{ixa} da + \frac{1}{\sqrt{2\pi}} \int_{2^j\pi}^{2^{j+1}\pi} 0 \cdot e^{ixa} da$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-2^{j+1}\pi}^{-2^j\pi} 0 \cdot e^{ixa} da$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-2^{j+1}\pi}^{2^{j+1}\pi} f(a) e^{ixa} da \text{ extended.}$$



Since $f(a) \equiv 0$ outside of $(-2^{j+1}\pi, 2^{j+1}\pi)$,

$f(x)$ (which ~~has not changed~~ has not changed) is in V_{j+1} .

$$\therefore V_j \subset V_{j+1}.$$

2 Density. What this says is that every $f \in \mathcal{L}^2$

can be approximated as closely as we wish by a function $f_j \in V_j$. ~~function~~

(4)

Separation, Look at $f \in V_j$ for $j << 0$. Then,
by using $\ell = -j$, we have

$$f(x) = \frac{1}{\sqrt{\pi}} \int_{-2^{-\ell}\pi}^{2^{-\ell}\pi} \hat{f}(k) e^{ikx} dk.$$

But this holds for all ℓ . This means that
 $\hat{f}(k) = 0$ for all $k \in [-2^{-\ell}\pi, 2^{-\ell}\pi]$ for all ℓ .

This ~~only~~ only can happen if happen if $\hat{f}(k) = 0$.
 $\Rightarrow f = 0$. Thus, separation holds.

Scaling Suppose $f(x) \in V_j$. Then,

$$f(x) = \frac{1}{\sqrt{\pi}} \int_{-2^j\pi}^{2^j\pi} \hat{f}(k) e^{ikx} dk$$

$$\Rightarrow f(2^{-j}x) = \frac{1}{\sqrt{\pi}} \int_{-2^j\pi}^{2^j\pi} \hat{f}(k) e^{2^{-j}kx} dk$$

Let $w = 2^{-j}k$. Then, $dw = 2^{-j}dk \Rightarrow dk = 2^j dw$

and

$$f(2^{-j}x) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \hat{f}(2^j w) e^{iwx} dw$$

$$f(2^{-j}x) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} (\hat{f}(2^j w) 2^j) e^{iwx} dw$$

$$\Rightarrow f[f(2^{-j}x)] = 0 \text{ outside of } [-\pi, \pi]$$

$$\Rightarrow f(2^{-j}x) \in V_0.$$

(5)

Also, from

Plancherel / Parseval Thm

(See section 2.2.4)

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R})}.$$

From property 6, rect. 2.2.1,

$$\mathcal{F}[f(x-a)] = e^{-ixa} \hat{f}(a).$$

Combining all of the above.

$$\begin{aligned} \int_{-\infty}^{\infty} \sin(c(x-h)) \sin(cx-l) dx &= \int_{-\infty}^{\infty} \mathcal{F}[\sin(c(x-h))] f[\sin(cx-l)] dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \overline{\int_{-\infty}^{\infty} \text{Box}(t) e^{-ict} dt} \cdot \overline{\int_{-\infty}^{\infty} \text{Box}(t) e^{-ilt} dt} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-icx} \cdot e^{+icx} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda(l-h)} d\lambda \stackrel{\downarrow}{=} \begin{cases} 1, & l=h \\ 0, & l \neq h \end{cases} = \delta_{h,l} \end{aligned}$$

Orthog. rels for Fourier series.

$$\therefore \int_{-\infty}^{\infty} \sin(c(x-h)) \sin(cx-l) dx = \delta_{h,l}.$$

$\therefore \left\{ \sin(c(x-h)) \right\}_{h=-\infty}^{\infty}$ is an orthonorm. set.

We'll skip the basis property (-i.e., span).