

Last time: MRA, Shannon & Haar MRAs.

Today: Wavelet relation & decomposing & reconstruction

1. Wavelet relation

(a) $W_j =$ Wavelet space for level $j = \{w \in V_{j+1} : w \perp \ell \in V_j\}$

It has a basis $\{\phi(2^j x - k)\}_{k=-\infty}^{\infty}$

where $\phi(2^j x - k) \in V_{j+1}$ and

$\leftarrow V_j \leftarrow W_j$

$$\langle \phi(2^j x - n), \phi(2^j x - m) \rangle = 0 \text{ for all } m \neq n.$$

~~Since this basis~~

(b) Wavelet relation: Since $\psi(x) \in V_1$, we can expand it in the V_0 basis:

$$\psi(x) = \sum_{k=-\infty}^{\infty} \langle \psi, \phi_{1,k} \rangle \phi_{1,k}(x)$$

$$\langle \psi, \phi_{1,k} \rangle = \int_{-\infty}^{\infty} \psi(x) \frac{1}{\sqrt{2}} \phi(2x - k) dx = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \psi(x) \phi(2x - k) dx$$

Define $g_k = 2 \int_{-\infty}^{\infty} \psi(x) \phi(2x - k) dx$. Then,

$$\langle \psi, \phi_{1,k} \rangle = \frac{1}{\sqrt{2}} g_k$$

$$\text{and } \psi(x) = \sum_{k=-\infty}^{\infty} \frac{g_k}{\sqrt{2}} \phi(2x - k) = \sum_{k=-\infty}^{\infty} g_k \phi(2x - k)$$

Wavelet relation:

$$\psi(x) = \sum_{k=-\infty}^{\infty} g_k \phi(2x - k), \quad g_k = 2 \int_{-\infty}^{\infty} \psi(x) \phi(2x - k) dx$$

Getting the g_n 's, since $\phi(x-n) \perp \psi(x) \forall n$,

we have $\langle \phi(x-n), \psi(x) \rangle = 0$. But,

$$\phi(x-n) = \sum_{k=-\infty}^{\infty} p_k \phi(2x-k-2n)$$

$$\phi(x-n) = \sum_{k=-\infty}^{\infty} p_k \phi(2x-(k+2n))$$

Let $k' = k+2n$, so

$$\phi(x-n) = \sum_{k'=-\infty}^{\infty} p_{k'-2n} \phi(2x-k')$$

$$\begin{aligned} \Rightarrow \langle \phi(x-n), \psi(x) \rangle &= \sum_{k'=-\infty}^{\infty} p_{k'-2n} \langle \phi(2x-k'), \psi(x) \rangle \\ &= \int_{-\infty}^{\infty} \phi(2x-k') \psi(x) dx \end{aligned}$$

Proof, $p_k = 2 \int_{-\infty}^{\infty} \phi(2x-k) \psi(x) dx$, so

$$\langle \phi(x-n), \psi(x) \rangle = \frac{1}{2} \sum_{k'=-\infty}^{\infty} p_{k'-2n} p_{k'} = 0$$

or, $\sum_{k=-\infty}^{\infty} p_{k-2n} p_k = 0$

Using the trick in the book, we can use

$$p_k = (-1)^k p_{-k}$$

Wavelet rel: $\psi(x) = \sum_{k=-\infty}^{\infty} (-1)^k p_{-k} \phi(2x-k)$

* Trick = reverse & change signs —

$$\begin{array}{r} 2 \quad 1 \quad -3 \quad -1 \\ -1 \quad 3 \quad 1 \quad -2 \\ \hline -2 + 3 - 3 + 2 = 0 \end{array}$$

~~$\begin{array}{r} 2 \quad 1 \quad -3 \quad -1 \\ -1 \quad 3 \quad 1 \quad -2 \end{array}$~~

~~$\begin{array}{r} 2 \quad 1 \quad -3 \quad -1 \\ -1 \quad 3 \quad 1 \quad -2 \end{array}$~~ we have:
 $+2 + 3 + 3 - 2 = 0$

2. Two bases.

2nd basis $\{ 2^{\frac{j}{2}} \phi(2^j x - k) \}_{k=-\infty}^{\infty} \cup \{ 2^{\frac{l}{2}} \psi(2^l x - k) \}_{k=-\infty}^{\infty}$

is an orthonormal basis for V_{j+l} . of course, so is $\{ 2^{\frac{j+l}{2}} \phi(2^{j+l} x - k) \}$, we already have

$$\left\{ \begin{aligned} \phi(x) &= \sum_{k=-\infty}^{\infty} p_k \phi(2^j x - k) \\ \psi(x) &= \sum_{k=-\infty}^{\infty} q_k \phi(2x - k), \quad q_k = (-1)^k p_{1-k} \end{aligned} \right.$$

We want $\phi(2x)$, $\phi(2x-m)$ ~~note~~ in terms of the 2nd basis.

$$\begin{aligned} \phi(2x-m) &= \sum_{k=-\infty}^{\infty} \langle \phi(2x-m), \phi(x-k) \rangle \phi(x-k) \\ &\quad + \sum_{k=-\infty}^{\infty} \langle \phi(2x-m), \psi(x-k) \rangle \psi(x-k) \end{aligned}$$

We ~~also~~ have $\langle \phi(2x-m), \phi(x-k) \rangle$

$$= \int_{-\infty}^{\infty} \phi(2t-m) \phi(t-k) dt$$

$$= \int_{t=k}^{t=k+1} \phi(2t-m) \phi(t) dt = \frac{1}{2} p_{m-2k}$$

Same extent

same calculation: $\langle \phi(2x-m), \psi(x-k) \rangle = \frac{1}{2} q_{m-2k}$

$$\Rightarrow \phi(2x-m)$$

$$= \sum_{k=-\infty}^{\infty} p_{m-2k} \phi(x-k) + \sum_{k=-\infty}^{\infty} q_{m-2k} \psi(x-k)$$

$$= \sum_{k=-\infty}^{\infty} p_{m-2k} \phi(2x-k) + \sum_{k=-\infty}^{\infty} (L^{-1})_{m-2k} \frac{\psi(2x-k)}{1+k-2k}$$

$$\phi(2x-m) = \sum_{k=-\infty}^{\infty} p_{m-2k} \phi(x-k) + \sum_{k=-\infty}^{\infty} q_{m-2k} \psi(x-k)$$

We work with the spaces V_0, W_0 and V_0 . Suppose that $f \in V_0$. Then, we have two expansions, one for each basis.

$$f(x) = \sum_{k=-\infty}^{\infty} a_k^1 \phi(2x-k) \quad (H1)$$

$$f(x) = \sum_{k=-\infty}^{\infty} a_k^0 \phi(x-k) + \sum_{k=-\infty}^{\infty} b_k^0 \psi(x-k) \quad (H2)$$

Decomposition. Using basis (H1), we have

~~from (H1),~~

$$\begin{aligned} a_m^0 &= \sum_{k=-\infty}^{\infty} a_k^1 \int_{-\infty}^{\infty} \phi(x-m) \phi(2x-k) dx + 0 \\ &= \int_{-\infty}^{\infty} \phi(t) \phi(2t - (k-2m)) dt \\ &= \int_{-\infty}^{\infty} \phi(t) \phi(2t - (k-2m)) dt \\ &= \frac{1}{2} p_{k-2m} \end{aligned}$$

$$\Rightarrow a_m^0 = \sum_{k=-\infty}^{\infty} \frac{1}{2} p_{k-2m} \phi(2x-k) \leftarrow a_k^1$$

Same calculation over using \mathcal{H} ,

$$\left. \begin{aligned}
 b_m^v &= \sum_{k=-\infty}^{\infty} \frac{1}{2} g_{k-2m} a_k^i \\
 a_m^v &= \sum_{k=-\infty}^{\infty} \frac{1}{2} l_{k-2m} a_k^i
 \end{aligned} \right\}$$

Filters $h_k = \frac{1}{2} g_{-k}, l_k = \frac{1}{2} g_{-k}$

$$\Rightarrow b_m^v = \sum_{k=-\infty}^{\infty} h_{2m-k} a_k^i = \text{~~(H * a^i)~~} \\
 = (h * a^i)_{2m}$$

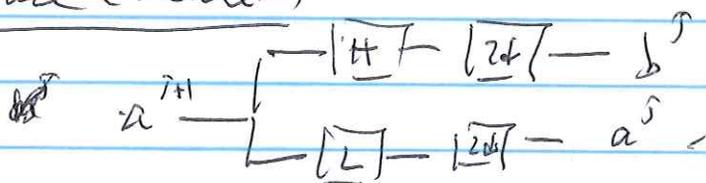
also $a_m^v = \sum_{k=-\infty}^{\infty} l_{2m-k} a_k^i = (l * a^i)_{2m}$

Using downsampling,

~~$$a_m^v = a_m^i$$~~

$$b^v = D H a^i, a^v = D L a^i$$

General case (scale!)



Reconstruction. We're working with V_0, W_0 and V_1 .
 The other cases are obtained by scaling.

Earlier:
$$f(2x-m) = \sum_{k=-a}^{\infty} p_{m-2k} f(x-k) + \sum_{k=-a}^a q_{m-2k} f(x-k)$$

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$$a_m^1 = \int_{-a}^{\infty} f(x) f(2x-m) dx = \sum_{k=-a}^{\infty} p_{m-2k} \int_{-a}^{\infty} f(x) f(x-k) dx + \sum_{k=-a}^a q_{m-2k} \int_{-a}^{\infty} f(x) f(x-k) dx$$

Now, $a_n^0 = \int_{-a}^{\infty} f(x) f(x-k) dx$, $b_n^0 = \int_{-a}^{\infty} f(x) f(x-k) dx$

$$\Rightarrow a_m^1 = \sum_{k=-a}^{\infty} p_{m-2k} a_n^0 + \sum_{k=-a}^a q_{m-2k} b_n^0$$

~~Consider the filters $\tilde{p}_k = p_k, \tilde{q}_k = q_k$. In terms of convolution we have~~

Let $\tilde{p}_k = p_k, \tilde{q}_k = q_k$. Then,

$$a_m^1 = \sum_{k=-a}^{\infty} \tilde{p}_{m-2k} a_n^0 + \sum_{k=-a}^a \tilde{q}_{m-2k} b_n^0$$

Upsample a^0 's & b^0 's,

$$a_m^1 = \sum_{k=-a}^{\infty} \tilde{p}_{m-k} (a^0)_k + \sum_{k=-a}^a \tilde{q}_{m-k} (b^0)_k$$

Filter form.

