Today: The Daubechies MRA.

1. Classification. Let \( P(z) = \frac{1}{2} \sum_{k=-a}^{a} P_n z^k \), \( a \) a Daubechies MRA, \( P(z) \) is a polynomial of degree \( 2N-1 \), \( N \geq 1 \). Then,

\[
P(z) = \frac{1}{2} (P_0 + P_1 z + \ldots + P_{2N-1} z^{2N-1}).
\]

Mallat's Conditions for an MRA. (General \( P(z) \).)

1. \( P(1) = 1 \)
2. \( |P(z)|^2 + |P(-z)|^2 = 1 \)
3. \( |P(e^{i\theta})| > 0 \) for \( \theta \in \mathbb{R} \).

If these hold, then the \( P_n \)'s are multiscale coefficients for an MRA.

Note that if we use \( z = 1 \) in (1), then \( |P(1)|^2 + |P(-1)|^2 = 1 \), so \( P(-1) = 0 \). Thus, \( P \) has a root at \( z = -1 \). Daubechies \( z \cdot P(z) \) has a root (not zero) of order \( N \):

\[
P(z) = (1 + z)^N \tilde{P}(z), \quad \tilde{P}(-1) = 0
\]

degree \( N-1 \),

For example, if \( N = 1 \), \( P(z) = (2 + 1)^z(2z+1) \). In fact, for
In fact, for $N=2$, the series of the coefficients in $P(z)$ are
\[ p_0 = \frac{1 + \frac{1}{\sqrt{3}}}{4}, \quad p_1 = \frac{3 + \frac{1}{\sqrt{3}}}{4}, \quad p_2 = \frac{3 - \frac{1}{\sqrt{3}}}{4}, \quad p_3 = \frac{3 - \frac{1}{\sqrt{3}}}{4}. \]

Moreover,
\[ P(z) = \left( \frac{1 + \frac{1}{\sqrt{3}}}{8} + \frac{3 + \frac{1}{\sqrt{3}}}{8} z + \frac{3 - \frac{1}{\sqrt{3}}}{8} z^2 + \frac{3 - \frac{1}{\sqrt{3}}}{8} z^3 \right). \]

2. "Moments." We will work with $N=2$. Recall that
\[ \Phi(3) = -e^{-\frac{z^2}{2}} P\left( -e^{\frac{1}{\sqrt{2} z}} \right) \Phi(2). \]

If we set $z=0$, then we get
\[ \Phi(0) = -P(-1) \Phi(0) = 0. \]

Since $\Phi(3) = \frac{1}{4\sqrt{\pi}} \int_0^\infty e^{-z^2} \Phi(x) dx$, we have
\[ \Phi(0) = C = \frac{1}{4\sqrt{\pi}} \int_0^\infty \Phi(x) dx. \] This is called the zeroth moment. Next, look at
\[ \Phi'(-3) = \frac{1}{2} e^{-\frac{z^2}{2}} P\left( -e^{\frac{1}{\sqrt{2} z}} \right) \Phi(2). \]

Let $z=0$, then
\[ \Phi'(0) = \frac{1}{2} \Phi(-1) \Phi(0) + \frac{1}{2} \Phi(-1) \Phi(0) - \Phi(-1) \Phi(0). \]
Since each \( \phi \) is even, because \( \text{but } P(-1) = 0 \) and \( P'(-1) = 0 \), we have

\[
\hat{\phi}'(0) = 0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(x) \hat{\psi}(x) dx,
\]

so

\[
\int_{-\infty}^{\infty} \hat{\psi}(x) \hat{\psi}(x) dx = 0. \text{ This is the "first" moment.}
\]

Now, in \( N=2 \), we have

\[
\int_{-\infty}^{\infty} \hat{\psi}(x) \hat{\psi}(x) dx = 0 \text{ and } \int_{-\infty}^{\infty} \hat{\psi}(x) dx = 0.
\]

3. Wavelet coefficients. It is possible to show that \( \text{supp } (\psi) = [-\infty, \infty] \) that is, \( \psi(x) = 0 \text{ if } x \notin [-\infty, \infty] \). The wavelets acts at level \( j \), the wavelet coefficients for a function \( f(x) \) are

\[
b^j_k = 2^j \int_{-\infty}^{\infty} f(x) \psi(2^{-j}x-k) dx,
\]

where \( k \in \mathbb{Z} \), the integers. Let's change the variables to \( t = 2^{j}x-k \);

\[
b^j_k = \int_{-\infty}^{\infty} f(t) \psi(t) dt = \int_{-\infty}^{\infty} f(t) \psi(t) dt.
\]

We can approximate \( f \) by

\[
f(2^{-j}x) \approx f(2^{-j}x + 2^{-j}t) f(2^{-j}x) + 2^{-j}t f'(2^{-j}x) + x 2^{-j/2} f''(2^{-j}x).
\]
Now, if \( j > 1 \),
\[
\beta^j \propto \sum_{n} 2^{-j/2} \left( f(2^{-j} n) + 2^{-j} f(2^{-j} n + 1) \right) + 2^{-j-1} \int (2^{-j} n)^2 \hat{g}(\tau) d\tau
\]

\[
\Rightarrow b^j = \left( \sum_{n} 2^{-j} \hat{g}(n) \right) \cdot f(2^{-j} n)
\]

\[
+ 2^{-j} \hat{g}(2^{-j} n) \int_{n-1}^{n} g(\xi) d\xi
\]

\[
+ 2^{-j-1} \hat{g}(2^{-j} n) \int_{n}^{n+1} \xi \hat{g}(\xi) d\xi
\]

Because both \( \int_{n}^{n+1} \xi \hat{g}(\xi) d\xi \) and \( \int_{n-1}^{n} g(\xi) d\xi \) are 0, we have

\[
b^j \propto 2^{-j-1} \hat{g}(2^{-j} n) \int_{n}^{n+1} \xi \hat{g}(\xi) d\xi, \quad j > 1
\]

**Implications**

Singularity detection: If \( f \) has a discontinuity in its first derivative, the approximation above fails, and \( b^j \) becomes too large relative to nearby coefficients. This fact allows one to look at the wavelet. That is, the corresponding wavelet coefficient will "jump," and give the location of the singularity.

Noise removal: When noise is present, the formula for \( b^j \) doesn't hold. The coefficients approximating \( f \) become large. Noise removal amounts to discarding "artificial" sum of the \( b^j \)'s.