Math 484
Apr. 27, 2020

Today: The Daubechies MRA.

1. Classification. Let $P(z) = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_k z^k$. For a Daubechies MRA, $P(z)$ is a polynomial of degree $2N-1$, $N \geq 1$. That is,

$$P(z) = \frac{1}{2}\left(p_0 + p_1 z + \cdots + p_{2N-1} z^{2N-1}\right).$$

Mallat's Conditions for an MRA. (General $P(z)$.)

$$\begin{cases} (1) & P(1) = 1 \\ (2) & |P(z)|^2 + |P(-z)|^2 \\ (3) & |P(e^{it})| > 0 \text{ for all } |t| \leq \pi/2. \end{cases}$$

If these hold, then the p_k's are the scaling coefficients for an MRA.

Note that if we use $z=1$ in (2), then $\underbrace{|P(1)|^2}_{1} + |P(-1)|^2 = 1$, so $P(-1) = 0$. Thus, P has at a root at $z=-1$. Daubechies: $P(z)$ has a root (a zero) of order N:

$$P(z) = (1+z)^N \underbrace{\tilde{P}(z)}_{\text{degree } N-1}, \quad \tilde{P}(-1) \neq 0$$

For example, if $N=2$, $P(z) = (z+1)^2(az+b)$. ~~In fact, for N~~

In fact, for $N=2$, ~~the coefs~~ the coefs. in $P(z)$ are

$$p_0 = \frac{1+\sqrt{3}}{4}, \quad p_1 = \frac{3+\sqrt{3}}{4}, \quad p_2 = \frac{3-\sqrt{3}}{4},$$

$$p_3 = \frac{3-\sqrt{3}}{4},$$

$$P(z) = \left(\frac{1+\sqrt{3}}{8}\right) + \left(\frac{3+\sqrt{3}}{8}\right)z + \left(\frac{3-\sqrt{3}}{8}\right)z^2 + \left(\frac{3-\sqrt{3}}{8}\right)z^3.$$

2. "<u>Moments</u>." We will work with $N=2$. Recall that

$$\hat{\psi}(\xi) = -e^{-\frac{i\xi}{2}} P\left(-e^{\frac{i\xi}{2}}\right) \hat{\phi}\left(\frac{\xi}{2}\right).$$

If we set $\xi = 0$, then we get

$$\hat{\psi}(0) = -P(-1)\,\hat{\phi}(0) = 0.$$

Since $\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \phi(x)\,dx$, we have

$$\hat{\psi}(0) = 0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\,dx.$$ This is called

the order 0 moment. Next, look at

$$\hat{\psi}'(\xi) = +\frac{i}{2} e^{-\frac{i\xi}{2}} P\left(-e^{\frac{i\xi}{2}}\right) \hat{\phi}\left(\frac{\xi}{2}\right)$$
$$- e^{-\frac{i\xi}{2}} \left(-\frac{i}{2}\right) P'\left(-e^{\frac{i\xi}{2}}\right) \hat{\phi}\left(\frac{\xi}{2}\right)$$
$$- e^{-\frac{i\xi}{2}} P\left(-e^{i\xi/2}\right) \cdot \frac{1}{2} \hat{\phi}'\left(\frac{\xi}{2}\right).$$

Let $\xi = 0$. $\hat{\psi}'(0) = \frac{i}{2} \overbrace{P(-1)}^{=0} \hat{\phi}(0) + \frac{i}{2} \overbrace{P'(-1)}^{=0} \hat{\phi}(0) - P(-1)\hat{\phi}'(0).$

Since each term is 0, because both $P(-1)=0$ and $P'(-1)=0$, we have

$$\hat{\psi}'(0) = 0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ix\,\psi(x)\,dx,$$

so $\int_{-\infty}^{\infty} x\,\psi(x)\,dx = 0$. This is the "first" moment.

Thus, for $N=2$, we have

$$\int_{-\infty}^{\infty} x\,\psi(x)\,dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \psi(x)\,dx.$$

3. <u>Wavelet coefficients</u> It is possible to show that $\operatorname{supp}(\psi) = [-1,2]$ — that is, $\psi(x) = 0$ if $x \notin [-1,2]$. ~~The wavelet coefs~~ At level j, the wavelet coefficients for a function $f(x)$ are

$$b_k^j = 2^j \int_{-\infty}^{\infty} f(x)\,\psi(2^j x - k)\,dx,$$

where $k \in \mathbb{Z}$ — the integers. Let's change ~~variable~~ variables ~~to~~ to $t = 2^j x - k$:

$$b_k^j = \int_{-1}^{2} f(2^{-j}k + 2^{-j}t)\,\psi(t)\,dt.$$

We can approximate f by

$$f(2^{-j}k + 2^{-j}t) \approx f(2^{-j}k) + 2^{-j}t \cdot f'(2^{-j}k) + 2^{-2j-1}t^2 \cdot f''(2^{-j}k).$$

Now, if $j \gg 1$,

$$b_k^j \approx \int_{-1}^{2} \Big(f(2^{-j}k) + 2^{-j} f'(2^{-j}k)\, t + 2^{-2j-1} f''(2^{-j}k)\, t^2 \ldots \Big) \psi(t)\, dt$$

$$\Rightarrow b_k^j \approx \left(\int_{-1}^{2} \psi(t)\, dt \right) \cdot f(2^{-j}k)$$
$$+ 2^{-j} f'(2^{-j}k) \int_{-1}^{2} t\, \psi(t)\, dt$$
$$+ 2^{-2j-1} f''(2^{-j}k) \cdot \int_{-1}^{2} t^2 \psi(t)\, dt$$

Because both $\int_{-1}^{2} \psi(t)\, dt$ and $\int_{-1}^{2} t\psi(t)\, dt$ are 0, we have

$$b_k^j \approx 2^{-2j-1} f''(2^{-j}k) \int_{-1}^{2} t^2 \psi(t)\, dt, \quad j \gg 1.$$

<u>Implications</u>

<u>Singularity detection</u>: If f has a discontinuity in its first derivative, the approximation above fails, and b_k^j become ~~[illegible] large and the~~ large relative to nearby coefficients. ~~This fact allows one to look at the wavelet~~ That is, the corresponding wavelet coefficient will "pop," and give the location of the singularity.

<u>Noise removal</u>. When noise is present, the formula for b_k^j doesn't hold. The coefficients approximating [illegible] become large. Noise removal amounts to discarding "artfully" some of the b_k^j's.