Midterm

This midterm should be emailed to me by 11:59 pm on Thursday, April 9, 2020. You may consult any written or online source. You may also consult your instructor; however, you may *not* consult anyone else.

1. **(20 pts.)** Problem 5.2.4. (Hint: regard the mass m the center of the string as contributing a delta function $m\delta(x-c)$, where c is the center of the string.)

2. A planet moving around the Sun in an elliptical orbit, with eccentricity $0 < \varepsilon < 1$ and period P, has time and angle (position) related in the following way. Let $\tau = (2\pi/P)(t - t_p)$, where t_p is the time when the planet is at perihelion – i.e., it is nearest the Sun. Let θ be the usual polar angle and let u be an angle related to θ via

$$(1-\varepsilon)^{1/2}\tan(u/2) = (1+\varepsilon)^{1/2}\tan(\theta/2).$$

It turns out that $\tau = u - \varepsilon\sin(u)$. All three variables θ, u, and τ are measured in radians. They are called the *eccentric*, and *mean anomalies*, respectively. (*Anomaly* is another word for angle.)

   (a) **(10 pts.)** Show that one may uniquely solve $\tau = u - \varepsilon\sin(u)$ for $u = u(\tau)$, that u is an odd function of τ, and that $g(\tau) = u(\tau) - \tau$ is an odd, 2π periodic function of τ.

   (b) **(10 pts.)** Because g is odd and 2π periodic, it can be represented by a Fourier sine series,

$$g(\tau) = \sum_{n=1}^{\infty} b_n \sin(n\tau).$$

   Show that $b_n = \frac{2}{n}J_n(\varepsilon)$, $n = 1, 2, \cdots$, where J_n is the n^{th} order Bessel function of the first kind. Thus, we have that $u = \tau + \sum_{n=1}^{\infty}(2/n)J_n(n\varepsilon)\sin(n\tau)$. (Hint: you will need to use equation (6.35) in the text.)

3. **(20 pts.)** Show that, for $n \ge 1$, $\frac{\Gamma'(n+1+z)}{\Gamma(n+1+z)} = \sum_{k=1}^{n} \frac{1}{k+z} + \frac{\Gamma'(1+z)}{\Gamma(1+z)}$. Specifically, for $z = 0$, $\frac{\Gamma'(n+1)}{\Gamma(n+1)} = \sum_{k=1}^{n} \frac{1}{k} - \gamma$, where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant. (One can show that $\gamma := \lim_{n\to\infty}\big(\sum_{j=1}^{n}\frac{1}{j} -$

$\log(n)\big) \approx 0.5772.$) Use this formula to obtain

$$Y_0(z) = \frac{2}{\pi}\big(\gamma + \log\big(\frac{z}{2}\big)\big)J_0(z) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\big(\frac{z}{2}\big)^{2n}\big(\sum_{k=1}^{n}\frac{1}{k}\big).$$

4. **(20 pts.)** Consider the Hermite polynomials[1],

$$H_n(x) := (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}).$$

Show that $e^{2tz-t^2} = \sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n$ is a generating function for the H_n's. Hint: $\frac{d^n}{dz^n}(e^{-z^2}) = \frac{n!}{2\pi i}\oint_C \frac{e^{-\zeta^2}}{(\zeta-z)^{n+1}}d\zeta$, where C is any simple closed contour containing z in its interior.

5. **(20 pts.)** The following is a special case of the Paley-Wiener Theorem. Let $f(z)$ be an entire function that satisfies these conditions: (1) for $x \in \mathbb{R}$, $f(x) \in L^1(\mathbb{R})$; (2) there exist constants $A > 0$, $\rho > 0$, and $\delta > 0$ such that $|f(z)| \le A(|z|+1)^{-\delta}e^{\rho|\mathrm{Im}(z)|}$ for all $z \in \mathbb{C}$. Show that for all $\xi \in \mathbb{R}$ such that $|\xi| > \rho$ one has that

$$\int_{-\infty}^{\infty} f(x)e^{i\xi x}dx = 0.$$

[1]The Hermite polynomials here are the ones that are used in physics in connection with the Harmonic oscillator potential, kx^2.