

Math 612
3/25/2020

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Last time: Bessel Functions of the first kind.

Bessel's Eqn.:

$$(*) \quad z^2 y'' + z y' + (z^2 - \nu^2) y = 0.$$

The solution to (*) for all ~~$\nu \in \mathbb{Z}$~~ $\nu \neq -1, -2, \dots$ is given by

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{\Gamma(\nu+1+k) \Gamma(k+1)}.$$

Wronskian

$$W(J_{-\nu}, J_\nu) = \frac{2}{\pi} \frac{\sin(\pi \nu)}{z}.$$

As long as $\nu \notin \mathbb{Z}$, the solns. J_ν and $J_{-\nu}$ are linearly independent.

$J_{-n}(z)$ for $n \in \{1, 2, 3, \dots\}$

~~Last time~~ $J_{-n}(z) = (-1)^n J_n(z)$

When $\nu = \text{integer}$,
 J_ν & $J_{-\nu}$
are linearly
dependent.

Today: A second solution for $\nu = 0, 1, 2, \dots$,

Recall that

$$W(\bar{J}_{-\nu}, \bar{J}_{\nu}) = \frac{2}{\pi} \frac{\sin(\pi\nu)}{\pi z},$$

so

$$W(\bar{J}_{\nu}, \bar{J}_{-\nu}) = -\frac{2}{\pi} \frac{\sin(\pi\nu)}{\pi z}.$$

1) For any $a \in \mathbb{C}$, we have

$$\boxed{W(\bar{J}_{\nu}, a\bar{J}_{-\nu} - \bar{J}_{\nu}) = \frac{2}{\pi z} \sin(\pi\nu)}$$

$$W(\bar{J}_{\nu}, \frac{a\bar{J}_{-\nu} - \bar{J}_{\nu}}{\sin(\pi\nu)}) = \frac{2}{\pi z}$$

(Recall that $W(\bar{J}_{\nu}, \bar{J}_{\nu}) = 0$.)

Let $a := \cos(\pi\nu)$ and set

$$(\dagger) \quad y_{\nu}(z) = \frac{\cos(\pi\nu)\bar{J}_{-\nu}(z) - \bar{J}_{\nu}(z)}{\sin(\pi\nu)}.$$

As long as $\nu \notin \mathbb{Z}$, $W(\bar{J}_{\nu}, y_{\nu}) = \frac{2}{\pi z} \neq 0$,
and y_{ν} is again a second solution
to Bessel's eqn.

The function $Y_\nu(z)$ is called a Neumann (or Weber) function - or a Bessel function of the 2nd kind.

2) Second solution for $\nu = 0, 1, 2, \dots$

It helps to define the function

$$F(z, \nu) = \cos(\pi\nu) J_\nu(z).$$

Look at

$$\begin{aligned} G(z, \nu) &:= \cos(\pi\nu) J_\nu(z) - J_{-\nu}(z) \\ &= \cos(\pi\nu) F(z, \nu) - F(z, -\nu), \end{aligned}$$

$$\begin{aligned} \text{Then, } \frac{\partial}{\partial \nu} [\cos(\pi\nu) F(z, \nu) - F(z, -\nu)] \\ &= -\pi \sin(\pi\nu) F(z, \nu) + \cos(\pi\nu) \frac{\partial F}{\partial \nu}(z, \nu) \\ &\quad - \frac{\partial}{\partial \nu} [F(z, -\nu)] \\ &= -\pi \sin(\pi\nu) F(z, \nu) + \cos(\pi\nu) \frac{\partial F}{\partial \nu}(z, \nu) \\ &\quad + \frac{\partial F}{\partial \nu}(z, -\nu), \quad \} \text{ Entire in } \nu. \end{aligned}$$

let $\nu \rightarrow n$.

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Let $\nu \rightarrow n$, $n = 0, 1, 2, \dots$. Then,

$$\frac{\partial G}{\partial \nu}(z, n) = 0 \cdot F(z, n) + (-1)^n \frac{\partial F}{\partial \nu}(z, n) + \frac{\partial F}{\partial \nu}(z, -n).$$

Now, it turns out (nasty calculation!) that $\frac{\partial G}{\partial \nu}(z, n) \neq 0$. Finally,

~~now~~

$$\begin{aligned} & \lim_{\nu \rightarrow n} \left(\frac{\cos(\pi \nu) J_\nu(z) - J_{-\nu}(z)}{\sin(\pi \nu)} \right) \\ & \stackrel{\uparrow}{=} \frac{(-1)^n \frac{\partial J_\nu}{\partial \nu} \Big|_{\nu=n} + \frac{\partial J}{\partial \nu} \Big|_{\nu=-n}}{\pi (-1)^n} \\ & \text{L'Hôpital.} \\ & \stackrel{\approx}{=} \frac{\frac{\partial J_\nu}{\partial \nu} \Big|_{\nu=n} + (-1)^n \frac{\partial J}{\partial \nu} \Big|_{\nu=-n}}{\pi} \\ & =: \frac{1}{\pi} Y(z). \end{aligned}$$

Types of Bessel Functions

1st kind: $J_\nu(z)$

2nd kind: $Y_\nu(z)$

Hankel Functions @ Bessel Functⁿ of the third kind

$$\begin{cases} H_\nu^{(1)}(z) = J_\nu(z) + i Y_\nu(z), \text{ Hankel of 1st kind} \\ H_\nu^{(2)}(z) = J_\nu(z) - i Y_\nu(z), \text{ Hankel of 2nd kind.} \end{cases}$$

Modified Bessel's Eqn $z \rightarrow it$

$$x^2 y''(x) + x y'(x) - (x^2 + \nu^2) y(x) = 0$$

Solutions

$J_\nu(it), J_\nu(-it)$ — BUT

$$I_\nu(z) = e^{\frac{-i\nu\pi}{2}} J_\nu(iz) \quad \left. \begin{array}{l} \text{Modified} \\ \text{Bessel of 1st$$

$$K_\nu(z) = \frac{1}{2} i\pi e^{\frac{i\nu\pi}{2}} H_\nu^{(1)}(iz) \quad \left. \begin{array}{l} \text{Modified} \\ \text{Bessel of} \\ \text{2nd kind.} \end{array} \right\}$$

Asymptotic Behavior for large $|z|$

We will only list the behavior when $z = x \gg 0$,
 $x \rightarrow \infty$.

$$\left. \begin{aligned}
 J_\nu(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\
 Y_\nu(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\
 H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\
 &\sim \left(\frac{2}{\pi x}\right)^{1/2} e^{i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)} \\
 H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) \\
 &\sim \left(\frac{2}{\pi x}\right)^{1/2} e^{-i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)} \\
 I_\nu(x) &\sim \frac{e^x}{\sqrt{2\pi x}} \\
 K_\nu(x) &\sim \sqrt{\frac{\pi}{2x}} e^{-x}
 \end{aligned} \right\}$$

Special Identities

$$(i) \frac{d}{dz} [z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z)$$

Proof: $\frac{d}{dz} [z^\nu \underbrace{J_\nu(z)}_{\sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{\Gamma(k+\nu+1)\Gamma(k+1)}}]$

$$= \frac{d}{dz} \left[\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+\nu}}{\Gamma(k+\nu+1)\Gamma(k+1)} z^\nu \right]$$

$$= \frac{d}{dz} \left[\sum_{k=0}^{\infty} \frac{(-1)^k z^{-2k-\nu} z^{2k+2\nu}}{\Gamma(k+\nu+1)\Gamma(k+1)} \right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{-2k-\nu} (2k+2\nu) z^{2k+2\nu-1}}{\Gamma(k+\nu+1)\Gamma(k+1)}$$

$= (k+\nu) \Gamma(k+\nu) \xrightarrow{-(2k+\nu-1)}$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cancel{2}^{-(2k+\nu-1)} z^{2k+2\nu-1}}{\Gamma(k+\nu)\Gamma(k+1)}$$

$$= z^\nu \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+\nu-1}}{\Gamma(k+\nu)\Gamma(k+1)} \left[z^{2k+2\nu-1} z^{-\nu} \right]$$

$$= z^\nu J_{\nu-1}(z)$$

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$$(ii) \quad \frac{d}{dz} \left[z^{-\nu} J_{\nu}(z) \right] \\ = -z^{-\nu} J_{\nu+1}(z)$$

The proof is pretty much the same,

The Generating Function

$$\Phi(z, t) := \sum_{n=-\infty}^{\infty} J_n(z) t^n$$

Go back to the identities (i) & (ii). For (i),

$$\frac{d}{dz} (z^{\nu} J_{\nu}) = J_{\nu-1}(z) - z^{\nu}$$

$$\Rightarrow z^{\nu} J_{\nu}' + \nu z^{\nu-1} J_{\nu} = J_{\nu-1}(z) - z^{\nu}$$

$$\text{For (ii), } \frac{d}{dz} (z^{-\nu} J_{\nu}) = -z^{-\nu} J_{\nu+1}(z)$$

$$\Rightarrow z^{-\nu} J_{\nu}' - \nu z^{-\nu-1} J_{\nu} = -z^{-\nu} J_{\nu+1}(z)$$

~~1/2~~ We can solve this for $J_{\nu}'(z)$:

$$J_{\nu}'(z) = \frac{1}{2} (J_{\nu-1}(z) - J_{\nu+1}(z))$$

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Now, we can use this to calculate $\frac{\partial \bar{\Phi}}{\partial z}$.

$$\frac{\partial \bar{\Phi}}{\partial z} = \sum_{n=-\infty}^{\infty} \frac{dJ_n}{dz} t^n$$

$$\frac{\partial \bar{\Phi}}{\partial z} = \sum_{n=-\infty}^{\infty} \frac{1}{2} (J_{n-1} - J_{n+1}) t^n$$

$$\frac{\partial \bar{\Phi}}{\partial z} = \sum_{\substack{k=-\infty \\ k=n-1}}^{\infty} \frac{1}{2} J_k t^{k+1} - \sum_{\substack{k=-\infty \\ k=n+1}}^{\infty} \frac{1}{2} J_k t^{k-1}$$

$$\frac{\partial \bar{\Phi}}{\partial z} = \frac{1}{2} t \cdot \bar{\Phi} - \frac{1}{2} t^{-1} \bar{\Phi}$$

$$\frac{\partial \bar{\Phi}}{\partial z} = \frac{1}{2} (t - t^{-1}) \bar{\Phi} \leftarrow \text{1st order ODE in } z$$

$$\Rightarrow \bar{\Phi}(z, t) = \underbrace{\bar{\Phi}(0, t)}_{=1} e^{\frac{1}{2}(t-t^{-1})z}$$

$$\Rightarrow \boxed{\bar{\Phi}(z, t) = e^{\frac{1}{2}(t-t^{-1})z}}$$

Integral Representations for $J_n(z)$.

Note that

$$\Phi(z, t) = \sum_{n=-\infty}^{\infty} J_n(z) t^n = e^{\frac{1}{2}(t-t^{-1})z}$$

is a Laurent series in t , valid in a neighborhood of $t=0$. The Laurent coef. here is $J_n(z)$.

$$\therefore J_n(z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{\Phi(z, t)}{t^{n+1}} dt$$

$$\Rightarrow \text{If } t = e^{i\theta}, dt = ie^{i\theta} d\theta \text{ and } t^{n+1} = e^{i(n+1)\theta},$$

with we have:

$$\left\{ \begin{aligned} J_n(z) &= \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{1}{2}z(e^{i\theta} - e^{-i\theta}) - in\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{iz \sin\theta - in\theta} d\theta. \end{aligned} \right.$$

Other, similar formulae can be derived.