

Math 642
Mar. 27, 2020

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Last time: We derived the 2nd soln., ~~$J_\nu(z)$~~ of Bessel's eqn., & obtained the three special identities —

$$\left\{ \begin{array}{l} \frac{d}{dz}(z^{-\nu} J_\nu) = -z^{-\nu} J_{\nu+1} \\ \frac{d}{dz}(z^\nu J_\nu) = z^\nu J_{\nu-1} \\ J_\nu'(z) = J_{\nu-1} - J_{\nu+1} \end{array} \right.$$

and derived the generating function for integer Bessel functions:

$$J(z, t) = e^{\frac{1}{2}(t-t^{-1})z} \sum_{n=-\infty}^{\infty} J_n(z) t^n,$$

formulas

This led to integral formulas for $J_n(z)$:

$$J_n(z) = \frac{1}{2\pi i} \oint e^{\frac{1}{2}(t-t^{-1})z} \frac{t^{-n-1}}{t^n} dt,$$

$$\text{if } \Rightarrow J_n(z) = \frac{1}{2\pi i} \int_0^{2\pi} e^{\frac{1}{2}(e^{it}-e^{-it})z} e^{-(n+1)i\theta} e^{iz\sin(\theta)} i e^{it} dt$$

$$\therefore J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin(\theta)-in\theta} dt.$$

~~(cont)~~ Can get: $J_n(z) = \frac{1}{\pi} \int_0^\pi u \sin(z \sin(u) - n u) du$

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Today: Generating function for Legendre polynomials.

1. Legendre polys.

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} [(z^2 - 1)^n].$$

Cauchy Integral Formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}} ds,$$

If we choose $f(z) = \frac{(z^2 - 1)^n}{z^n}$,

then

then we get

$$\frac{d^n}{dz^n} \left[\frac{(z^2 - 1)^n}{z^n} \right] = \frac{n!}{2\pi i} \int_C \frac{(z^2 - 1)^n}{z^n (z - z)^{n+1}} dz$$

$$\approx \frac{1}{2^n n!} \frac{d^n}{dz^n} [(z^2 - 1)^n] = \frac{1}{2\pi i} \int_C \frac{(z^2 - 1)^n}{z^n (z - z)^{n+1}} dz,$$

$$\boxed{P_n(z) = \frac{1}{2\pi i} \int_C \frac{(z^2 - 1)^n}{z^n (z - z)^{n+1}} dz}$$

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Define: $\phi(z, t) := \sum_{n=0}^{\infty} P_n(z) t^n.$

? want to show that

$$\phi(z, t) = \frac{1}{1 - 2zt + t^2}.$$

of: By the integral formula for $P_n(z)$,

$$\phi(z, t) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(\bar{z}-1)^n z^n}{2^n (\bar{z}-z)^{n+1}} d\bar{z} \quad \left. \begin{array}{l} \text{Use} \\ |\bar{z}| = 1 \\ \text{as } C. \end{array} \right\}$$

$$\rightarrow \phi(z, t) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \left[\frac{(\bar{z}-1)t}{2(\bar{z}-z)} \right]^n \frac{d\bar{z}}{\bar{z}-z} \quad \left. \begin{array}{l} \\ \text{C} \end{array} \right\}$$

$$= \frac{1}{2\pi i} \int_C \underbrace{\sum_{n=0}^{\infty} \left(\frac{(\bar{z}-1)t}{2(\bar{z}-z)} \right)^n}_{\text{Geometric series.}} \frac{d\bar{z}}{\bar{z}-z}$$

→ Converges for $|t|$ small enough.

$$\phi(z, t) = \frac{1}{2\pi i} \int_C \left(\frac{1}{1 - \frac{(\bar{z}-1)t}{2(\bar{z}-z)}} \right) \frac{d\bar{z}}{\bar{z}-z}.$$

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$$\oint_C \frac{d\bar{z}}{\bar{z}^2 - (\bar{z}^2 - 1)t}$$

Algebra

$$\oint_C \frac{d\bar{z}}{\bar{z}^2 t - 2\bar{z} + 2z - t}$$

Mare Algebra

Quad. Polynomial in \bar{z} .

Roots: $\bar{z}_\pm = \frac{2 \pm \sqrt{4 - 4t(2z-t)}}{2t}$

$$\bar{z}_\pm = \frac{1 \pm \sqrt{t^2 - 2zt + 1}}{t}$$

Small t , \bar{z}_- is inside, so

$$\oint_C \frac{d\bar{z}}{\bar{z}^2 t - 2\bar{z} + 1} = \frac{1}{1 - 2zt + t^2},$$

Also, $P_n(z) = \frac{1}{2\pi i} \oint_C \frac{d\bar{z}}{\bar{z}^{n+1} / (1 - 2z\bar{z} + \bar{z}^2)},$

2 Unbounded Operators

Let H be a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

Defⁿ Let D be a subspace of H and let \tilde{L} be a linear transformation that maps $D \rightarrow H$. We say that L is a linear operator on H with domain D . (Throughout, we will assume that $D(L) = \text{domain } L$ is dense in H .)

Remarks.

- (i) Closed operator L is closed if and only if this holds: If $f_n \rightarrow f$ and $Lf_n \rightarrow g$, then $f \in D(L)$ and $Lf = g$.
→ An unbounded operator being closed is similar to continuity is similar to L being continuous.

(ii) Extensions, An operator \tilde{L} is said to be an extension of L iff $D(\tilde{L}) \supseteq D(L)$ and $\tilde{L}|_{D(L)} = L$. We write $\tilde{L} \supseteq L$.

(iii) L is closable iff it has a closed extension.

~~Example~~ Examples of unbounded ops: Differential ops,

$\leftarrow L$

Adjoint Operator

with domain

Let L be densely defined, on $D(L)$.

Suppose that $g \in \mathcal{H}$ has the property that there is a $\tilde{g} \in \mathcal{H}$ for which

$$(*) \quad \langle Lf, g \rangle = \langle f, \tilde{g} \rangle \quad \forall f \in D(L),$$

(i) \tilde{g} is uniquely determined by g .

Proof: Suppose that \tilde{g}_1 & \tilde{g}_2 satisfy (*).

Then, $\langle Lf, g \rangle = \langle f, \tilde{g}_1 \rangle$ & $\langle Lf, g \rangle = \langle f, \tilde{g}_2 \rangle$

if

$$\langle f, \tilde{g}_1 \rangle = \langle f, \tilde{g}_2 \rangle$$

if

$$(*) \quad \langle f, \tilde{g}_1 - \tilde{g}_2 \rangle = 0 \quad \forall f \in D(L)$$

Since $D(L)$ is dense in \mathcal{H} , taking limits implies that (*) holds for all $f \in \mathcal{H}$.

Choose $f = \tilde{g}_1 - \tilde{g}_2$. (*) $\Rightarrow \| \tilde{g}_1 - \tilde{g}_2 \|^2 = 0$
 $\Rightarrow \tilde{g}_1 = \tilde{g}_2$.

Consequence, the correspondence $g \mapsto \tilde{g}$ defines a function on \mathcal{H} .

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(ii) The correspondence is linear.

Proof: Let g_1, g_2 correspond to \tilde{g}_1, \tilde{g}_2 ,

respectively. Then,

$$\langle Lf, g_1 + g_2 \rangle = \langle f, \tilde{g}_1 \rangle + \langle f, \tilde{g}_2 \rangle$$

$$= \langle f, \tilde{g}_1 + \tilde{g}_2 \rangle$$

$$= \langle f, g_1 + g_2 \rangle.$$

$$\Rightarrow g_1 + g_2 \leftrightarrow \tilde{g}_1 + \tilde{g}_2.$$

Also, $\alpha g \leftrightarrow \alpha \tilde{g}$.

Defⁿ, let L be densely defined. Then, we define L^* the adjoint of L , to be the operator that takes $g \rightarrow \tilde{g}$. Thus, $\langle Lf, g \rangle = \langle f, L^*g \rangle$.

The domain of L^* is the set of all g s.t. $\exists \tilde{g}$ for which $\langle Lf, g \rangle = \langle f, L^*g \rangle$.

Proposition. L^* is closed. (Assume $D(L)$ is dense)

Proof: Let $\{g_n\} \subset D(L^*)$ and suppose $g_n \rightarrow g$.
In addition, suppose $L^*g_n = h_n \rightarrow h$. Then,

$$\langle L^*f, g_n \rangle = \langle f, L^*g_n \rangle = \langle f, h_n \rangle$$



$$\langle L^*f, g \rangle = \langle f, h \rangle.$$

$\therefore g \in D(L^*)$ and $L^*g = h$. By defⁿ,

L^* is closed.

Proposition. L^* , the adjoint of a densely defined operator L , is densely defined if and only if L has a closed extension.
[Proof: see Riesz-Nagy].

Self-Adjoint operators. We say that L is self-adjoint iff $L = L^*$. (That is, $D(L) = D(L^*)$ and $L = L^*$.)

Remark. (i) Since adjoints are always closed, a self-adj. op. L is closed.

(ii) Then, every closed linear translat. s, t : $D(L) = \mathbb{H}$ is bounded.

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Example. Let $H = L^2(\mathbb{R})$ and let

$$Lf = xf \quad \text{w/ } D(L) = \{f \in H : \int_{-\infty}^{\infty} |x|^2 |f|^2 dx < \infty\}.$$

First of all, ~~D(L)~~ L is linear: If $xf \in L^2$ and $xg \in L^2$, then, of course, ~~xf + pg~~

$$\alpha xf + \beta xg \in L^2.$$

$\Rightarrow D(L)$ is a subspace of H .

Second, $D(L)$ is dense in H .

- Exercise,

Third, L is closed,

~~To show~~: If $Lf_n = g_n$ is such that both $f_n \rightarrow f$ and $g_n \rightarrow g$, then

$$\int_{-\infty}^{\infty} |xf_n - g_n|^2 dx \rightarrow 0$$

$$\text{hence } \int_{-\infty}^{\infty} |f_n - f|^2 dx \rightarrow 0.$$

On $L^2[-R, R]$, we have $\int_R^R |xf_n - g_n|^2 dx$

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$$\| \chi f_n - g \|_{H_K} = \| \chi (f_n - f) + \cancel{\chi f - f} \|_{H_K}$$

if

$$\| \chi f_n - g \| \geq \left(\| \chi f - g \|_{H_K} + R \| f_n - f \|_{H_K} \right)$$

$$\Rightarrow \| \chi f - g \|_{H_K} \leq \| \chi f_n - g \|_{H_K} + R \| f_n - f \|_{H_K}$$

fixed $\rightarrow \| \chi f - g \|_{H_K} \leq \| \chi f_n - f \|_{L^2(\mathbb{R})}$

$$+ R \| g_n - g \|_{L^2(\mathbb{R})},$$

Let $n \rightarrow \infty$, R fixed.

$$\Rightarrow \| \chi f - g \|_{H_K} = 0,$$

BUT, this holds for all R . Hence

$$\| \chi f - g \|_{L^2(\mathbb{R})} = 0$$

$$\Rightarrow \chi f = g$$

$$\therefore L(f) = g$$

$\Rightarrow L$ is closed.