Math 642
Apr. 1, 2020

Last time: Spectrum: $\sigma(L) = \rho(L)^{c}$; $\sigma = \sigma_d + \sigma_r + \sigma_c$;
$\rho(L) = \{ \lambda \in \mathbb{C} : (L - \lambda I)^{-1} \in \underbrace{B(\mathcal{H})}_{\text{Bounded ops. on } \mathcal{H}.} \}$

Today: Operator-valued analytic fnct.

I. Operator-val. analytic fnct.

Def^n: Let $\Omega \in \mathbb{C}$, where Ω is open in $\mathbb{C}$. Then, we say at $L : \Omega \to B(\mathcal{H})$ is analytic at $z \in \Omega$ iff

$$\lim_{h \to 0} \frac{L(z+h) - L(z)}{h} = \frac{dL}{dz}$$

exists at every ~~pt~~ ~~point~~ point in Ω. Note: the limit above is in the operator norm:

$$\lim_{h \to 0} \left\| \frac{L(z+h) - L(z)}{h} - \frac{dL}{dz} \right\| = 0.$$

Proposition. $L(z)$ is analytic in Ω iff $\forall f \in \mathcal{H}$

$$F(z) = \langle L(z) f, f \rangle$$

is a scalar-valued fn. of z s.t. F is analytic on Ω.

Proof: ~~Let~~ See T. Kato, p. 138.

<u>Theorem:</u> The resolvent set $\rho(L)$ is open and the resolvent operator

$$R_\lambda(L) = (L - \lambda I)^{-1}$$

is analytic in $\rho(L)$.

<u>Proof</u>: Suppose $\lambda_0 \in \rho(L)$. Consider

$$(*) \quad L - \lambda I = L - \lambda_0 I - (\lambda - \lambda_0) I .$$

First of all, all of these operators map $D(L)$ into $\mathcal{H}$. Consequently, since $(L - \lambda_0 I)^{-1} \in B(\mathcal{H})$, $(L - \lambda_0 I)^{-1} = R_{\lambda_0} \in B(\mathcal{H})$. Then, $(L - \lambda_0 I)^{-1} : \mathcal{H} \to D(L)$.

Next, $R_{\lambda_0}(L - \lambda I)$ maps $D(L)$ to $D(L)$.

We thus have from $(*)$

$$(\dagger) \quad R_{\lambda_0}(L - \lambda I) = I - (\lambda - \lambda_0) R_{\lambda_0}$$

Suppose $|\lambda - \lambda_0| < \dfrac{1}{\|R_{\lambda_0}\|}$. Then we have

$$(I - (\lambda - \lambda_0) R_{\lambda_0})^{-1} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}^k ,$$

which follows from the Neumann exp. for $|\lambda - \lambda_0| < \dfrac{1}{\|R_{\lambda_0}\|}$. Note that, because

$(I-(\lambda-\lambda_0)R_{\lambda_0})^{-1}$ is a ~~power~~ power series in λ, it is analytic in λ, $|\lambda-\lambda_0|<\frac{1}{\|R_{\lambda_0}\|}$. Thus

$$(I-(\lambda-\lambda_0)R_{\lambda_0})^{-1}R_{\lambda_0}(L-\lambda I)=I,$$

which at this point only holds for $f\in D(L)$.

By look at (*) & multiplying on the right, ~~with~~ by R_λ, we have

$$(L-\lambda I)R_{\lambda_0}=I-(\lambda-\lambda_0)R_{\lambda_0}.$$

$$\Rightarrow \quad (L-\lambda I)R_{\lambda_0}(I-(\lambda-\lambda_0)R_{\lambda_0})^{-1}=I,$$

which holds for all $f\in\mathcal{H}$. ~~From this.~~
From these, we obtain

$$R_\lambda=(L-\lambda I)^{-1}=R_{\lambda_0}(I-(\lambda-\lambda_0)R_{\lambda_0})^{-1},$$

which holds for all $|\lambda-\lambda_0|<\frac{1}{\|R_{\lambda_0}\|}$. It follows ~~that R_λ is analytic on~~ that R_λ is analytic in $|\lambda-\lambda_0|<\frac{1}{\|R_{\lambda_0}\|}$. It also follows that if $|\lambda-\lambda_0|<\frac{1}{\|R_{\lambda_0}\|}$, $\lambda\in\rho(L)$.

<u>Cor.</u> $\sigma(L)=\rho(L)^c$ is closed.

~~Statement.~~ The complement of an open set is closed.

2. The first resolvent identity.

Prop. Let $\lambda, \lambda' \in \rho(L)$, then

$$R_\lambda - R_{\lambda'} = (\lambda - \lambda') R_\lambda R_{\lambda'}$$

Proof: $(L - \lambda I)(R_\lambda - R_{\lambda'}) = I - (L - \lambda I) R_{\lambda'}$

$$= I - (\cancel{L - \lambda' I + \lambda'} \; (L - \lambda' I + (\lambda' - \lambda) I) R_{\lambda'}$$

$$= I - (L - \lambda' I) R_{\lambda'} - (\lambda' - \lambda) R_{\lambda'}$$

$$= I - I + (\lambda - \lambda') R_{\lambda'}$$

$$(L - \lambda I)(R_\lambda - R_{\lambda'}) = (\lambda - \lambda') R_{\lambda'}$$

$$\Rightarrow \underbrace{R_\lambda (L - \lambda I)}_{I} (R_\lambda - R_{\lambda'}) = (\lambda - \lambda') R_\lambda R_{\lambda'}$$

$$\Rightarrow R_\lambda - R_{\lambda'} = (\lambda - \lambda') R_\lambda R_{\lambda'}.$$

Cor. $R_\lambda R_{\lambda'} = R_{\lambda'} R_\lambda$. $\longleftarrow$ $\cancel{R_{\lambda'}}$

Proof: ~~Interchange~~ $\lambda' \leftrightarrow \lambda \Rightarrow R_{\lambda'} - R_\lambda = (\lambda' - \lambda) R_{\lambda'} R_\lambda$

$$\Rightarrow R_\lambda - R_{\lambda'} = \underbrace{(-1)(\lambda' - \lambda)}_{(\lambda - \lambda')} R_{\lambda'} R_\lambda = (\lambda - \lambda') R_{\lambda' \lambda} = (\lambda - \lambda') R_{\lambda'} R_\lambda$$