Last line: Spectrum: \( \sigma(L) = \rho(L)^\frac{1}{2} \); \( \sigma = \sigma_\text{ess} + \sigma_0 + \sigma_\text{i} \)
\[
\rho(L) = \int \lambda \in \sigma : (L - \lambda I)^{-1} \in B(H)^{-1} \mathbb{C}
\]

Today: Operator-valued analytic funct.

\section{Operator-valued analytic funct.}

Defn: Let \( \Omega \subset \mathbb{C} \) open, \( \Omega \) is analytic at \( z \in \Omega \) \( L: \Omega \rightarrow \mathcal{B}(H) \) is analytic at \( z \in \Omega \) if

\[
\lim_{h \to 0} \frac{L(z+h) - L(z)}{h} = \frac{dL}{dz}
\]

exists at every \( z \) near point in \( \Omega \). Note: The limit above is in the operator norm:

\[
\lim_{h \to 0} \| \frac{L(z+h) - L(z)}{h} \| \frac{dL}{dz} \| = 0.
\]

Proposition: \( L(z) \) is analytic in \( \Omega \) iff \( \forall F \in \mathcal{L} \)

\[
L(z) = \langle L(z) F, F \rangle
\]

is a scalar-valued fn. of \( z \) st. \( F \) is analytic on \( \Omega \).

Proof: \( \text{see T. Kato, p. 138.} \)
Theorem: The resolvent set $\sigma(L)$ is open and the resolvent operator

$$R_\lambda(L) = (L - \lambda I)^{-1}$$

is analytic in $\sigma(L)$.

Proof: Suppose $\lambda_0 \in \sigma(L)$. Consider

$$(I) \quad L - \lambda_0 I = R_{\lambda_0} L - (L - \lambda_0 I).$$

First of all, all of these operators map $D(L)$ into $D(L)$. Consequently, since $(L - \lambda_0 I)^{-1} \in \mathcal{B}(A)$, $(L - \lambda_0 I)^{-1} = R_{\lambda_0} \mathcal{B}(A)$, then $(L - \lambda_0 I)^{-1} : D(L) \to D(L)$.

Next, $R_{\lambda_0} (L - \lambda_0 I)$ maps $D(L)$ to $D(L)$.

We then have from $(I)$

$$(II) \quad R_{\lambda_0} (L - \lambda_0 I) = I - (L - \lambda_0 I) R_{\lambda_0}.$$ 

Suppose $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$. Then we have

$$(I - (\lambda - \lambda_0) R_{\lambda_0})^{-1} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}^k,$$

which follows from the Neumann series for $|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$. Note that because
\[(I - (a - \lambda_0) R_{\lambda_0}^{-1})^{-1} \text{ power series in } \lambda, \text{ it is} \]
\[\text{analytic in } A, \quad |a - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|} \]
\[
(I - (a - \lambda_0) R_{\lambda_0}^{-1}) R_{\lambda_0}^{-1} (I - 2 \lambda I) = I,
\]

which at no point does it hold for \( \lambda \in \mathbb{D}(\delta_{1,0}) \).

By looking at (*) and multiplying on the right by \( R_{\lambda_0} \), we have
\[
(L - \lambda I) R_{\lambda_0}^{-1} = (L - \lambda I) R_{\lambda_0}^{-1} \]
\[
= (L - \lambda I) R_{\lambda_0}^{-1} (I - (a - \lambda_0) R_{\lambda_0}^{-1})^{-1} = I,
\]

which holds for all \( \lambda \in \mathbb{D} \). From this,

From here, we obtain
\[
R_{\lambda} = (L - \lambda I)^{-1} R_{\lambda_0} (I - (a - \lambda_0) R_{\lambda_0}^{-1}),
\]

which holds for all \( |a - \lambda| < \frac{1}{\|R_{\lambda_0}\|} \). It follows that \( R_{\lambda} \) is analytic in \( A \).

Also, from the above, it follows that if \( |a - \lambda| < \frac{1}{\|R_{\lambda_0}\|} \), \( \lambda \in \mathbb{D}(\delta_{1,0}) \),

\[
\text{Conv. } \sigma(\Lambda) = \rho(\Lambda)^c \text{ is closed.}
\]

\[\text{The complement of an open set is closed.} \]
2. The Fredholm resolvent identity.

Proposition. Let $a, a' \in \mathbb{R}$, then

$R_a - R_{a'} = (a - a') R_a R_{a'}$

Proof:

$(a - a') R_a R_{a'} = (a - a') R_a R_{a'}$

$= 1 - (a - a') R_{a'}$

$= 1 - (a - a') R_{a'}$

$(a - a') R_a R_{a'} = (a - a') R_a R_{a'}$

$= R_a (a - a') R_a R_{a'}$

$= R_a R_{a'} - R_{a'}$

$= R_a R_{a'} - R_{a'}$

$= R_a R_{a'} - R_{a'}$

conclude:

$R_a R_{a'} = R_a R_{a'}$

Proof:

Enters $a' = a$, then

$R_a - R_a = (a - a') R_a R_{a'}$

$= R_a - R_a = (a - a') R_a R_{a'}$

$= R_a R_{a'} = (a - a') R_a R_{a'}$

$= R_a R_{a'} = (a - a') R_a R_{a'}$

$= R_a R_{a'} = (a - a') R_a R_{a'}$

$= R_a R_{a'} = (a - a') R_a R_{a'}$