Math 642
Apr. 13, 2020

Last time: spectral measures — an example.
Today: Spectral measures — continued.

1. An Example

$$L = -\frac{d^2}{dx^2}, \quad D(L) = \{ u \in L^2 : u'' \in L^2 \text{ and } u(0) = u(1) = 0 \}$$

Spectrum $\{ k^2\pi^2 \}_{k=1}^{\infty}$.

Green's function for the resolvent.

$$\begin{cases} LG - \lambda G = \delta(x-y) \\ G(0,y) = 0 \\ G(1,y) = 0 \end{cases}$$

Solution. $G(x,y,\lambda) = -\dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda})} \begin{cases} \sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}(1-y)) \\ \sin(\sqrt{\lambda}(1-x))\sin(\sqrt{\lambda}y) \end{cases}$

The Resolvent:

$$(L - \lambda I)^{-1} f = \int_0^1 \underbrace{G(x,y,\lambda)}_{\text{Resolvent kernel.}} f(y).$$

Projections $\sigma = \{ \pi^2, 4\pi^2, \dots k^2\pi^2, \dots n^2\pi^2 \}$, (

$$\underbrace{P_\sigma(x,y)}_{\text{Kernel}} = 2 \sum_{k=1}^{n} \sin(k\pi x)\sin(k\pi y).$$

$$P_\sigma f = \sum_{k=1}^{n} 2\sin(k\pi x)\left(\int_0^1 \sin(k\pi y) f(y)\,dy\right)$$

$$\begin{cases} P_\sigma f = \sum_{k=1}^{n} b_k \sin(k\pi x), \\ b_k = 2\int_0^1 \sin(k\pi y) f(y)\,dy. \end{cases}$$

$$\Rightarrow \underbrace{P_\sigma f(x) = f_n(x) = \sum_{k=1}^{n} b_k \sin(k\pi x)}_{\text{① } f_n \text{ is the partial sum of the sine series for } f.}.$$

② $P_\sigma = \mathrm{Proj}_{V_n}$,

$V_n = \{\sin(\pi x), \sin(2\pi x), \cdots \sin(n\pi x)\}$.

~~2. Use the spectral family for L.~~

2. <u>The spectral family for L.</u>

Let $\begin{cases} P_k(x,y) = 2\sin(\pi k x)\sin(\pi k y) \\ P_k : \text{Projects onto the eigenspace of } L \text{ corresponding to } \lambda_k = \pi^2 k^2. \end{cases}$

$$\Rightarrow P_\sigma f = \sum_{k=1}^{n} P_k f.$$

<u>Strong Continuity</u> A sequence of operators T_k is said to converge strongly to an operator T if and only if for all $f \in \mathcal{H}$,

$$\lim_{k\to\infty} T_k f = Tf.$$

(1) If $\lim_{k\to\infty} \|T - T_k\| = 0$ — i.e., continuous in the norm of $\mathcal{B}(\mathcal{H})$, then $\{T_k\}_{k=1}^{\infty}$ is also strongly continuous.

<u>Proof</u>: Let $f \in \mathcal{H}$, then

$$\|(T - T_k)f\|_{\mathcal{H}} \leq \|T - T_k\|_{\mathcal{B}(\mathcal{H})} \longrightarrow 0.$$

$\therefore$ $\{T_k\}$ is strongly continuous.

(2) A strongly convergent sequence may fail to be convergent in the norm of $\mathcal{B}(\mathcal{H})$.

<u>Proof</u>: Consider the sum

$$P_\sigma = \sum_{k=1}^{n} P_k, \quad \sigma = \{1, \cdots, n^2\pi^2\}$$

We have $P_\sigma f(x) = P_n(x) = \sum_{k=1}^{n} b_k \sin(k\pi x)$.

This is a sine series. If $f \in L^2$, then

$\lim_{n\to\infty} \|P_n - f\| = 0$, ~~[illegible]~~

~~no~~ because the set $\{\sin(k\pi x)\}_{k=1}^{\infty}$ is a complete orthogonal set. Look at ~~this~~ this ~~map~~ in terms of projections:

$$P_\sigma = P_{\{\pi^2, 4\pi^2 \dots n^2\pi^2\}} = \sum_{k=1}^{n} P_k$$

Since P_σ is a projection, we have that

$$\|I - P_\sigma\|^2 = \sup_{\substack{f \in \mathcal{H} \\ \|f\|=1}} \|(I-P_\sigma)f\|^2.$$

Choose $f \perp \{\sin(\pi x), \dots, \sin(n\pi x)\}$. Then,

$$P_\sigma f = 0, \quad \|f\|^2 = 1$$

$$\Rightarrow \|(I-P_\sigma)f\|^2 = \|f\|^2 = 1.$$

$$\therefore \|I - P_\sigma\|_{op} = 1,$$

Thus, even though $P_\sigma \to I$ strongly, it does not converge to I in the operator norm.

Prop ~~Prop~~ The project E_λ is strongly right continuous.

Proof: On any interval of the form $[n^2\pi^2, (n+1)^2\pi^2)$, E_λ is constant. Hence, $s\text{-}\lim_{\varepsilon \downarrow 0} E_{\lambda+\varepsilon} = E_\lambda$.

Proposition Let $\lambda' < \lambda$. Then,

$$\underbrace{E_{\lambda'} E_\lambda}_{} = \underbrace{E_\lambda E_{\lambda'}}_{} = E_{\lambda'}$$

These projections commute

Proof: If λ, λ' are in an interval $[n^2\pi^2, (n+1)^2\pi^2)$, then, as noted above, $E_\lambda = E_{\lambda'} = \sum_{k=1}^{n} P_k$, so $E_\lambda E_{\lambda'} = E_{\lambda'}^2 = E_{\lambda'}$. (Of ~~course~~ course, in this case, $E_\lambda E_{\lambda'} = E_\lambda$.) Suppose that $m < n$ and ~~$\lambda' \pi^2$~~ $\lambda' \in [m^2\pi^2, (m+1)^2\pi^2)$ and $\lambda \in [n^2\pi^2, (m+1)^2\pi^2)$. Then,

$$E_{\lambda'} = \sum_{k=1}^{m} P_k, \quad E_\lambda = \sum_{k=1}^{n} P_k.$$

Recall that the P_k's are ~~mutual projections~~

~~Recall that if $k \neq j$, $P_k P_j = 0$. Since $P_j^2 = P_j$, we may~~

Recall that $P_k P_j = 0$ if $j \neq k$. In particular, if $j > m$, then

$$P_j E_{\lambda'} = P_j \sum_{k=1}^{m} P_k = \sum_{k=1}^{m} \underbrace{P_j P_k}_{j \neq k,\ j > k} = 0.$$

~~Too~~ Write $$E_\lambda = \sum_{k=1}^{n} P_k = \sum_{k=1}^{m} P_k + \sum_{k=m+1}^{n} P_k.$$

$$\Rightarrow \quad E_\lambda = E_{\lambda'} + \sum_{j=m+1}^{n} P_j$$

$$E_{\lambda'} E_\lambda = E_{\lambda'}^2 + \sum_{j=m+1}^{n} \underbrace{E_{\lambda'} P_j}_{=0} = E_{\lambda'}^2 = E_{\lambda'}.$$

$$\therefore \quad E_{\lambda'} E_\lambda = E_{\lambda'}.$$

<u>Cor.</u> Let $\lambda' < \lambda$, then $E_\lambda - E_{\lambda'}$ is a projection.

<u>Proof</u>: $$(E_\lambda - E_{\lambda'})^2 = E_\lambda^2 - 2E_\lambda E_{\lambda'} + E_{\lambda'}^2$$

$$= E_\lambda - 2E_{\lambda'} + E_{\lambda'}$$

$$= E_\lambda - E_{\lambda'}.$$

The family of projections $\{E_\lambda\}$ satisfies the ~~fol~~ following properties:

(i) $E_\lambda E_{\lambda'} = E_{\lambda'}$ for $\lambda' \leq \lambda$.

(ii) $s\text{-}\lim_{\varepsilon \downarrow 0} E_{\lambda+\varepsilon} = E_\lambda$ (Right continuity)

(iii) $E_\lambda = 0$ if $\lambda < \pi^2$ and

$$s\text{-}\lim_{\lambda \to +\infty} E_\lambda = I$$

A family of projections satisfying these conditions is called a <u>spectral family</u>.

<u>Remark</u>. We can replace (i) with the condition that $E_{\lambda'} \leq E_\lambda$ — which means that for all $f \in \mathcal{H}$,

$$\langle E_{\lambda'} f, f \rangle \leq \langle E_\lambda f, f \rangle$$

<u>Proof</u>: Consider $\langle (E_\lambda - E_{\lambda'}) f, f \rangle$. Earlier, we showed that $(E_\lambda - E_{\lambda'})^2 = E_\lambda - E_{\lambda'}$. Then

$$\langle (E_\lambda - E_{\lambda'}) f, f \rangle = \langle (E_\lambda - E_{\lambda'})^2 f, f \rangle$$
$$= \langle (E_\lambda - E_{\lambda'}) f, \underbrace{(E_\lambda - E_{\lambda'})}_{\text{self-adj}} f \rangle \geq 0$$

We leave the converse as an exercise