Last time: Spectral measures — an example.

Today: Spectral measures — continued.

1. An Example

\[ L = -\frac{d^2}{dt^2}, \quad D(L) = \left\{ u \in L^2 : u'' \in L^2 \text{ and } u(0) = u(\pi) = 0 \right\} \]

Spectrum \( \left\{ \lambda_k^2 \right\}_{k=1}^{\infty} \).

Green’s function for the resolvent.

\[
\begin{cases}
  (L - \lambda I) G = \delta(x-y) \\
  G(0,y) = 0 \\
  G(1,y) = 0
\end{cases}
\]

Solution. \[ G(x,y) = -\frac{1}{12} \sin(x) \sin(\lambda_1 y) \left( \frac{\sin(\lambda_1 x) \sin(\lambda_1 y)}{\sin(\lambda_1)} \right) \]

The Resolvent.

\[ (L - \lambda I)^{-1} f = \int_0^1 G(x,y) f(y) \, dy, \]

resolvent kernel.

Projectors \( \sigma = \{ n^2, 4n^2, \ldots, k_1^2 \pi^2, \ldots, n^2 \pi^2 \} \),

\[ P_0 (x,y) = \sum_{k=1}^{n^2} \frac{1}{2} \sin(kx) \sin(ky). \]

Kernel
\[ P \phi = \sum_{k=1}^{\infty} a_k \sin(k\pi x) \left( \int_0^1 \sin(k\pi y) f(y) \,dy \right) \]

\[ P_{\phi} f = \sum_{k=1}^{\infty} b_k \sin(k\pi x) \]

\[ b_k = 2 \int_0^1 \sin(k\pi y) f(y) \,dy. \]

\[ P_{\phi} \phi = \sum_{k=1}^{\infty} b_k \sin(k\pi x) = \sum_{k=1}^{\infty} a_k \sin(k\pi x). \]

1. \( P_{\phi} \phi \) is the partial sum of the sine series for \( I \).
2. \( P_{\phi} f = \text{Proj}_V f \), where \( V = \{ \sin(\pi x), \sin(2\pi x), \ldots, \sin(n\pi x) \} \).

8. Assume the spectral family \( \{ P_k \} \).
2. The spectral family \( \{ P_k \} \).

Let \( P_k(x, y) = 2 \sin(k\pi x) \sin(k\pi y) \)

\[ P_k : \text{Projects onto the eigenspace of } L \text{ corresponding to } \lambda_k = k^2. \]

\[ P_{\phi} f = \sum_{k=1}^{\infty} P_k f. \]
Strong Continuity. A sequence of operators $T_k$ is said to converge strongly to an operator $T$ if and only if for all $f$,

$$\lim_{k \to \infty} T_k f = T f.$$

$(1)$ If $\lim_{k \to \infty} \|T_k f - h\| = 0$ – i.e., $T_k$ converges in the norm of $B(\mathcal{H})$, then $\{T_k\}_{k=1}^\infty$ is also strongly continuous.

**Proof:** Let $f \in \mathcal{H}$, then

$$\|T_k f - h\| \leq \|T_k f - T_k h\| + \|T_k h - h\| \to 0,$$

so $\{T_k\}_{k=1}^\infty$ is strongly continuous.

$(2)$ A strongly convergent sequence may fail to be convergent in the norm of $B(\mathcal{H})$.

**Proof:** Consider the sum

$$P_0 = \sum_{k=1}^n P_k, \quad \sigma = \{1, \ldots, n \}^2$$

We have $P_0 f(x) = \sum b_k \sin(\alpha x), \quad \alpha = \{1, \ldots, n \}^2$. This is a sine series. If $f = \sum_{k=1}^n f_k$ for

$$\lim_{n \to \infty} \|P_n f-k \| = 0,$$

then

$$\sum_{k=1}^n f_k = s.$$
because we have \( \sum_{k=1}^{\infty} |\sin(k \pi x)|^2 \) in a complete orthogonal set. Look at this more in terms of projections:

\[
P_0 = P_{\pi^2, \pi^2, \ldots, \pi^2} = \sum_{k=1}^{\infty} \frac{\pi^2}{k^2} P_k.
\]

Since \( P_0 \) is a projection, we have that

\[
\|I - P_0\|_2^2 = \langle f, (I - P_0)f \rangle = \sup_{\|f\|_2 = 1} \| (I - P_0)f \|_2^2 = 1.
\]

Choose \( f \in \sin(\pi x), \ldots, \sin(2m \pi x) \), then

\[
P_0 f = 0, \quad \|f\|_2^2 = 1
\]

\[
\Rightarrow \| (I - P_0)f \|_2^2 = \| f \|_2^2 = 1.
\]

\[
\Rightarrow \| I - P_0 \|_2 = 1.
\]

Thus, even though \( P_0 \rightarrow I \) strongly, it does not converge to \( I \) in the operator norm.
Proposition: Let $\lambda' < \lambda$. Then,

\[ E_{\lambda'} E_{\lambda} = E_{\lambda'} E_{\lambda} = E_{\lambda'} \]

Proof: If $\lambda, \lambda'$ are in an interval $\bigcup_{n} \left( n^2 \pi^2, (n+1)^2 \pi^2 \right)$, then, as noted above, $E_{\lambda} = E_{\lambda'} = \sum_{k=1}^{n} P_k$, so

\[ E_{\lambda'} E_{\lambda} = E_{\lambda'} \]

(If $\lambda' = \lambda$, case $E_{\lambda'} = E_{\lambda}$)

Suppose that $m < n$ and

$\lambda' \in \bigcup_{n} \left( m^2 \pi^2, (m+1)^2 \pi^2 \right)$ and

$\lambda \in \bigcup_{n} \left( m^2 \pi^2, (m+1)^2 \pi^2 \right)$. Then,

\[ E_{\lambda'} = \sum_{k=1}^{m} P_k \quad E_{\lambda} = \sum_{k=1}^{n} P_k \]

Recall that the $P_k$'s are...
Recall that if \( k \neq j \), then \( E_j E_k = 0 \). Hence \( \sum_j P_j = P \) and \( \sum_j P_j^2 = P^2 \).

Recall that \( \sum_k P_k = 0 \) if \( j \neq k \). In particular, if \( j, k \neq n \), then

\[
P_k^2 E_j = P_k \sum_{k=1}^n P_k^2 = \sum_{k=1}^n P_k^2 P_k = 0,
\]

for \( k \neq n \). So, we have:

\[
E_j = \sum_{k=1}^n P_k = \sum_{k=1}^{m} P_k + \frac{n}{m+1} P_n.
\]

Let

\[
\frac{E_j}{A} = E_{j}, \quad \frac{E_k}{A} = E_{k}, \quad \frac{E_j}{A} = E_{j}, \quad \frac{E_k}{A} = E_{k}, \quad \frac{E_j - E_k}{A} = E_{j-k}.
\]

Then

\[
E_j E_j = E_j^2 = \sum_{s=m+1}^n P_s E_j = E_j^2 = E_j.
\]

\[
E_j E_k = E_k = 0.
\]

\[
E_j - E_k = E_{j-k}.
\]

Cor. Let \( j < k \), then \( E_j - E_k \) is a projector.

Proof. \( (E_j - E_k)^2 = E_j^2 - 2E_j E_k + E_k^2 \)

\[
= E_j - 2E_j E_k + E_k
\]

\[
= E_j - E_k.
\]
The family of projectors \( \{ E_x \} \) satisfies the following properties:

(i) \( E_{\lambda} E_{\lambda'} = E_{\lambda} \) for \( \lambda' \leq \lambda \).

(ii) \( s\text{-lim}_{\lambda \to \infty} E_{\lambda} = E \) (right continuous).

(iii) \( E_{\lambda} = 0 \) if \( \lambda < \pi \) and
\[
\lim_{\lambda \to \pi^-} E_{\lambda} = I - E.
\]

A family of projectors satisfying these conditions is called a spectral family.

Remark. We can replace (i) with the condition that \( E_{\lambda} \leq E_{\lambda'} \), which means
\[
w \in \text{all } f, \quad < E_{\lambda} f, f > \leq < E_{\lambda'} f, f >.
\]

Proof. Consider \( < (E_{\lambda} - E_{\lambda'}) f, f > \). Earlier, we showed that \( (E_{\lambda} - E_{\lambda'})^2 = E_{\lambda} - E_{\lambda'} \). Then,
\[
< (E_{\lambda} - E_{\lambda'}) f, f > = < (E_{\lambda} - E_{\lambda'})^2 f, f > = < (E_{\lambda} - E_{\lambda'}) f, (E_{\lambda} - E_{\lambda'}) f > > 0
\]
We leave the converse as an exercise.