Last line:

\[ D(\epsilon) = \{ u \in L^2 \mid u(0) = u(1) = 0 \} \]

Eigenvalues:
\[ \pi^2, 4\pi^2, 9\pi^2, \ldots, k^2\pi^2, \ldots \]

Connection with sine series + projections:

\[ \sigma = \{ \pi^2, 4\pi^2, 9\pi^2, \ldots, k^2\pi^2 \} \]

\[ P_\sigma = \sum_{k=1}^{\infty} P_k, \quad \text{where} \quad P_k f = \frac{1}{\pi} \int_0^1 \sin(\pi k x) \phi(x) f(x) \, dx \]

\[ P_0 f = \sum_{k=1}^{\infty} b_k \sin(\pi k x) = f \quad \text{Partial sum} \]

Spectral projection:

\[ E_A f = \sum_{k=1}^{\infty} P_k f + (A - \lambda k), \quad H(t) = \begin{cases} 1, & 0 \leq t < 0 \\ 0, & t < 0 \end{cases} \]

Properties of \( E_A \):

(i) \( E_A E_A = E_A \), \( E_A^2 = E_A \), \( \lambda \leq 1 \)

(ii) \( \lim_{\lambda \to 0} E_A \phi = \hat{\phi} \) (right continuous)

(iii) \( \lim_{\lambda \to -\infty} E_A = 0, \quad \lim_{\lambda \to +\infty} E_A = 1 \)

Remark:

(i) \( \lambda \leq \lambda \) is equivalent to saying that \( E_A \leq E_A \)

\[ \lambda \leq \lambda \quad \iff \quad \langle E_A f, f \rangle \leq \langle E_A f, f \rangle \]
Today: Spectral measure, spectral resolution & spectral mean for $L^2$.

1. Spectral Measure

Recall that

$$E_j = \sum_{k=1}^{\infty} P_k H(\lambda - 2\pi_j),$$

and the spectral measure is just

$$dE_j = \sum_{k=1}^{\infty} P_k dH(\lambda - 2\pi_j).$$

Riemann-Stieltjes Integral. Let $\alpha(x)$ be of bounded variation on an interval $[a, b]$ and let $P = \{ a \leq x_0 < x_1 < \ldots < x_n = b \}$ be a partition of $[a, b]$. As usual, we define $||P|| = \max \{ x_k - x_{k-1} \}$. Then, $1 \leq n$.

The Riemann-Stieltjes integral is defined as

$$\int_a^b f(x) d\alpha(x) = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(x_k) \left( \alpha(x_k) - \alpha(x_{k-1}) \right),$$

where $x_{k-1} \leq \xi_k \leq x_k$ and $f(x)$ is assumed continuous on $[a, b]$. (Actually, we can get away with much weaker assumptions.)
Remark

If \( \alpha(x) \) is a step function in \( a < x < b \), with jumps at \( a < c_1 < \ldots < c_n < b \), then

\[
\int_a^b \alpha(x) \, dx = \sum_{k=1}^n \alpha(c_k^+)\, \Delta \alpha(c_k^-),
\]

If we allow jumps at \( a \) and \( b \), then

\[
\int_{a^-}^{b^+} \alpha(x) \, dx = \alpha(a^-) \Delta \alpha(a^+) + \sum_{k=1}^n \alpha(c_k^+) \Delta \alpha(c_k^-) + \alpha(b^+) \Delta \alpha(b^-),
\]

Resolution of the identity

\[
\left( \int_a^b \frac{dx}{x^2 + 1} \right) \varphi = \sum_{k=1}^n \frac{P_k \, \varphi}{\pi} \int_a^b \frac{dx}{x^2 + 1 - \lambda_k},
\]

\[
\int_a^b \frac{dx}{x^2 + 1} = \sum_{k=1}^n \frac{P_k}{\pi} \int_a^b \frac{dx}{x^2 + 1 - \lambda_k},
\]

\[
\varphi \in C^2[0,1]: \quad \int_a^b \frac{dx}{x^2 + 1} = \sum_{k=1}^n \frac{P_k \, \varphi}{\pi} \int_a^b \frac{dx}{x^2 + 1 - \lambda_k}.
\]
Resolution of the identity

Let \( f \in L^2[0,1] \),

then,

\[
\left( \int_{A_{\lambda}} d\xi \right) f = \sum_{k=1}^{\infty} \left( \int_{A_{\lambda}} d\xi \right) P_k f = \sum_{k=1}^{\infty} \left( \frac{1}{\lambda} \left( \frac{1}{1-(\lambda-2k)^2} - \frac{1}{1-(\lambda-2k+1)^2} \right) \right) \times P_k f = \sum_{k=1}^{\infty} P_k f
\]

Since \( P_k f = 2 \int_{-\lambda}^{\lambda} \sin((2\lambda-k)\pi y) f(y) dy \sin(k\pi x) \),

we have

\[
\left( \int_{A_{\lambda}} d\xi \right) f = \sum_{k=1}^{\infty} b_k \sin(k\pi x) = f,
\]

\[
\Rightarrow \int_{A_{\lambda}} d\xi = I.
\]

2. The spectral theorem (special case)

\[
\left( \int_{A_{\lambda}} d\xi \right) f = \sum_{k=1}^{\infty} \lambda_k P_k f = \sum_{k=1}^{\infty} P_k (lf) = \left( \sum_{k=1}^{\infty} P_k \right) lf = lf
\]

\[
\Rightarrow \int_{A_{\lambda}} d\xi = I.
\]
3. Spectral Family

Let $\mathcal{E}$ be a projection valued function of $\lambda \in \mathbb{R}$. We say that $\mathcal{E}$ is a spectral family of $\mathcal{A}$ if and only if the following hold:

\[ (\text{i}) \quad \frac{\mathcal{E}_\lambda \mathcal{E}_{\lambda'}}{\lambda'} = \mathcal{E}_\lambda \quad \text{if} \quad \lambda' \leq \lambda \]

\[ (\text{ii}) \quad \lim_{\Delta \to 0} \mathcal{E}_{\lambda + \Delta} = \mathcal{E}_\lambda \quad \text{(Right continuous)} \]

\[ (\text{iii}) \quad \lim_{\lambda \to -\infty} \mathcal{E}_\lambda = 0 \quad \text{and} \quad \lim_{\lambda \to +\infty} \mathcal{E}_\lambda = \mathcal{I}, \]

where $\mathcal{I}$ is the spectral measure of $\mathcal{A}$.

There is an equivalent (and useful) version of (i):

\[ (\text{i}') \quad \forall \lambda' \leq \lambda, \quad \mathcal{E}_{\lambda'} \leq \mathcal{E}_\lambda \quad \text{that is,} \]

\[ \forall f \in \mathcal{A}, \quad \langle \mathcal{E}_{\lambda'} f, f \rangle \leq \langle \mathcal{E}_\lambda f, f \rangle. \]

Moreover,

\[ \int_\mathbb{R} \mathcal{A} \, d\mathcal{E}_\lambda = \mathcal{I}. \]

**Proof:** Actually, this follows from the definition above. It is called a resolution of the identity.
4. Spectral Theorem. To every self-adjoint operator \( L \) in \( H \), there is a spectral family \( \{ E_\lambda \} \), which is uniquely determined \( L \), such that

\[
L = \int_{-\infty}^{\infty} \lambda \, dE_\lambda.
\]

We give a specific example of their form:

\[
L = -\frac{d^2}{dx^2},
\]

5. Spectral Transform. Let's return to our example, with \( L = -\frac{d^2}{dx^2} \), \( D(L) = \{ u \in L^2 : u'' \in L^2, u(0) = u(\pi) = 0 \} \)

\[
\langle \Phi, \int_0^\pi \sin(kx) \rangle = \langle \Phi, \sin(kx) \rangle
\]

\[
\sum_{k=1}^{\infty} \langle \Phi, \sin(kx) \rangle \sin(kx) \leq \langle \Phi, \Phi \rangle^{1/2} \leq \langle \Phi, \Phi \rangle^{1/2}
\]

There are two parts to this transform:

- Decomposition: \( \Phi \rightarrow \{ \langle \Phi, \sin(kx) \rangle \}
- Reconstruct: \( \sum \langle \Phi, \sin(kx) \rangle \sin(kx) \rightarrow \Phi \)
6. Obtaining $E_A$ and getting the spectral transform

Suppose that $L = L^*$ and that we know that a spectral family exists for $L$ but we don't know what $E_A$ is.

(i) Recall that if $c$ is an isolated part of the spectrum of $L$, then

$$P_c = \frac{-1}{2\pi i} \oint_C (L - \lambda I)^{-1} \, d\lambda,$$

is an orthogonal projection.

In our example, if $c = \{1\}$, we have

$$P_1 = \frac{-1}{2\pi i} \oint_C (L - 1) \, d\lambda,$$

where $C$ is

If the spectrum is continuous, $c$ is a mix of eigenvalues and continuous, then this doesn't quite work.
Proposition. Let \( \mu < 1 \) and define the "curve" \( \Gamma \) as shown:

Then,

\[
\frac{1}{2} [E(\lambda) + E(\lambda^-)] - \frac{1}{2} [E(\mu) + E(\mu^-)] = -\frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\Gamma} (z - \lambda)^{-1} dz,
\]

**Proof:** next time.