Math 642

Apr. 15, 2020

Last time:

$Lu = -u''$, $D(L) = \{u \in L^2 : u'' \in L^2, u(0) = u(1) = 0\}$

Eigenvalues $\pi^2, 4\pi^2, 9\pi^2, \dots k^2\pi^2, \dots$

Connection with sine series & projections

$\sigma = \{\pi^2, 4\pi^2, 9\pi^2, \dots, n^2\pi^2\}$

$$P_\sigma = \sum_{k=1}^{n} P_k, \quad \text{where} \quad P_k f = \underbrace{\left(2\int_0^1 \sin(k\pi y) f(y)\,dy\right)}_{b_k} \sin(k\pi x).$$

$$\Rightarrow \quad P_\sigma f = \sum_{k=1}^{n} b_k \sin(\pi k x) = f_n \leftarrow \text{partial sum of } \sum_{k=1}^{\infty} b_k \sin(k\pi x) = f$$

Spectral projections

$$E_\lambda f = \sum_{k=1}^{\infty} P_k H(\lambda - \lambda_k), \quad H(t) = \begin{cases} 1, & 0 \le t < \infty \\ 0, & t < 0, \end{cases}$$

Properties of E_λ
$$\begin{cases} \text{(i)} & E_{\lambda'} E_\lambda = E_\lambda E_{\lambda'} = E_{\lambda'}, \ \lambda' \le \lambda \\ \text{(ii)} & \lim_{\varepsilon \downarrow 0} E_{\lambda+\varepsilon} = E_\lambda \quad \text{(Right continuous)} \\ \text{(iii)} & \lim_{\lambda \to -\infty} E_\lambda = 0, \ \lim_{\lambda \to +\infty} E_\lambda = I \end{cases}$$

Remark: (i) is equivalent to saying that $E_{\lambda'} \le E_\lambda$ if $\lambda' \le \lambda$ — i.e. $\langle E_{\lambda'} f, f\rangle \le \langle E_\lambda f, f\rangle$

<u>Today</u>: spectral measure, spectral resolution & spectral theorem for L.

1. <u>Spectral Measure</u>

Recall that $E_\lambda = \sum_{k=1}^{\infty} P_k H(\lambda - \lambda_k)$.

The spectral measure is just

$$dE_\lambda = \sum_{k=1}^{\infty} P_k \, dH(\lambda - \lambda_k).$$

<u>Riemann-Stieltjes Integral</u> Let $\alpha(x)$ be of bounded variation on an interval $[a,b]$ and let $P = \{a \le x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. As usual, we define $\|P\| = \max_{1 \le k \le n} (x_k - x_{k-1})$. Then,

The ~~Stieltjes~~ Riemann Stieltjes integral is defined as

$$\int_a^b f(x)\,d\alpha(x) = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(\xi_k)\big(\alpha(x_k) - \alpha(x_{k-1})\big)$$

~~where $x_{k-1} \le \xi_k \le x_k$~~

~~where~~ where $x_{k-1} \le \xi_k \le x_k$ and $f(x)$ is assumed continuous on $[a,b]$. (Actually, we can get away with much weaker assumptions.)

Remark

If $\alpha(x)$ is a step function on $a<x<b$, with jumps at $a<c_1<\dots<c_n<b$, then

$$\int_a^b f(x)\,d\alpha(x) = \sum_{k=1}^{n} f(c_k)\underbrace{\left(\alpha(c_k^+)-\alpha(c_k^-)\right)}_{\text{jump}}.$$

If we allow jumps at a and b, then

$$\int_{a^-}^{b^+} f(x)\,d\alpha(x) = f(a)\left(\alpha(a^+)-\alpha(a^-)\right) + \sum_{k=1}^{n} f(c_k)\left(\alpha(c_k^+)-\alpha(c_k^-)\right) + f(b)\left(\alpha(b^+)-\alpha(b^-)\right).$$

~~Resolution of the identity~~

$$\cancel{\left(\int_{(\pi^2)^-}^{\infty} dE_\lambda\right) f = \sum_{k=1}^{\infty} P_k f \int_{\pi^2}^{\infty} dH(\lambda-\lambda_k)}$$

$$\cancel{= \sum_k}$$

Let $f\in L^2[a,b]$.

$$\left(\int_{\lambda_1^-}^{\infty} dE_\lambda\right) f = \sum_{k=1}^{\infty} (P_k f) \underbrace{\int_{\lambda_1^-}^{\infty} dH(\lambda-\lambda_k)}_{1}$$

$\cancel{H(\lambda^+ - \lambda_k)^+ H(\lambda-\lambda_k)^+} = 1$

<u>Resolution of the identity</u> Let $f \in L^2[0,1]$.

Then,
$$\left(\int_{\lambda_1^-}^{\infty} dE_\lambda\right) f = \sum_{k=1}^{\infty} \left(\int_{\lambda_1^-}^{\infty} d\mathcal{H}(\lambda-\lambda_k)\right) P_k f$$
$$= \sum_{k=1}^{\infty} \left(\overbrace{\mathcal{H}((\lambda-\lambda_k)^+)}^{1} - \overbrace{\mathcal{H}((\lambda-\lambda_k)^-)}^{0}\right) \times P_k f$$
$$= \sum_{k=1}^{\infty} P_k f$$

Since $P_k f = \underbrace{\left(2\int_0^1 \sin(\pi k y) f(y)\, dy\right)}_{b_k} \sin(\pi k x)$,

we have $\left(\int_{\lambda_1^-}^{\infty} dE_\lambda\right) f = \sum_{k=1}^{\infty} b_k \sin(\pi k x) = f.$

$$\Rightarrow \int_{\lambda_1^-}^{\infty} dE_\lambda = I.$$

2. <u>The spectral theorem</u> (special case)

$$\left(\int_{\lambda_1^-}^{\infty} \lambda\, dE_\lambda\right) f = \sum_{k=1}^{\infty} \lambda_k P_k f = \sum_{k=1}^{\infty} P_k (Lf)$$
$$= \underbrace{\left(\sum_{k=1}^{\infty} P_k\right)}_{I} Lf = Lf$$

$$\therefore \boxed{\int_{\lambda_1^-}^{\infty} \lambda\, dE_\lambda = L.}$$

3. Spectral Family

Let E_λ be a projection valued function of $\lambda \in \mathbb{R}$. We say that E_λ is a spectral family if and only if the following hold:

$$\begin{cases} \text{(i)} & E_{\lambda'} E_\lambda = E_{\lambda'} \quad \text{if } \lambda' \le \lambda \\ \text{(ii)} & \lim_{\varepsilon \downarrow 0} E_{\lambda+\varepsilon} = E_\lambda \quad \text{(Right continuous)} \end{cases}$$

$$\text{(iii)} \quad s\text{-}\lim_{\lambda \to -\infty} E_\lambda = 0 \text{ and } s\text{-}\lim_{\lambda \to +\infty} E_\lambda = I.$$

~~The spectral measure is dE_λ.~~

There is an equivalent (& useful) version of (i):

(i)' For $\lambda' \le \lambda$, $E_{\lambda'} \le E_\lambda$ — that is,

if $f \in \mathcal{H}$, $\langle E_{\lambda'} f, f \rangle \le \langle E_\lambda f, f \rangle$.

<u>Proposition</u> $\int_{-\infty}^{\infty} dE_\lambda = I$.

<u>Proof</u>: Actually, this follows from the definition above. It is called a resolution of the identity.

4. <u>Spectral Theorem</u> To every self-adjoint operator L on $\mathcal{H}$ there is a spectral family $\{E_\lambda\}$, which is uniquely determined by L, such that

$$L = \int_{-\infty}^{\infty} \lambda \, dE_\lambda .$$

We ~~stay~~ gave a specific example of this in $L = -\frac{d^2}{dx^2}$.

5. <u>Spectral Transform</u> Let's return to our example, with $L = -\frac{d^2}{dx^2}$, $D(L) = \{ u \in L^2 ; u'' \in L^2, u(0) = u(1) = 0 \}$

$$\left(\int_{-\infty}^{\infty} d E_\lambda \right) f = \sum_{k=1}^{\infty} \langle f, \sqrt{2} \sin(\pi k y) \rangle \sqrt{2} \sin(\pi k x) \cancel{\langle f(y), \sin(k\pi y) \rangle \sin(k\pi x)}.$$

There are two parts to this transform:

- <u>Decomposition</u> $f \to \{ \langle f(x), \sqrt{2} \sin(\pi k y) \}$
- <u>Reconstruction</u>

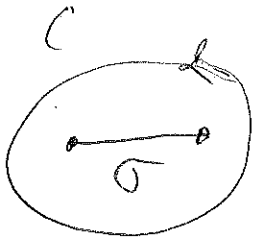
$$\{ \langle f(y), \sqrt{2} \sin(\pi k x) \rangle \} \longrightarrow f.$$

6. <u>Obtaining E_λ & getting the spectral transform</u>

Suppose that ~~[illegible]~~ $L = L^*$ and that we <u>know</u> that a spectral family exists for L, but we don't know what $\{E_\lambda\}$ is.

(1) Recall that if σ is an isolated part of the spectrum of L, then

$$P_\sigma = -\frac{1}{2\pi i}\oint_C (L-\lambda I)^{-1}\,d\lambda,$$

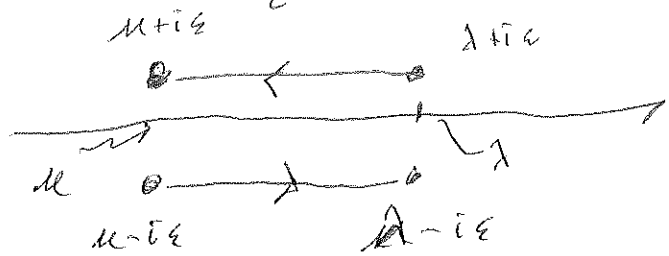


is an orthogonal projection.

~~Lit~~ In our example, if $\sigma = \{\lambda_n\}$, we had $P_n = -\frac{1}{2\pi i}\oint_C (L-\lambda I)^{-1}\,d\lambda$, where C is

. If the spectrum is continuous, or is a mix of eigenvalues and continuous, then this doesn't quite work.

<u>Proposition.</u> Let $\mu < \lambda$ and define the "curve" Γ_ε as shown:



Then,

$$\frac{1}{2}\left(E(\lambda)+E(\lambda^-)\right)-\frac{1}{2}\left(E(\mu)+E(\mu^-)\right)$$

$$= -\frac{1}{2\pi i}\lim_{\varepsilon\downarrow 0}\int_{\Gamma_\varepsilon}(L-\zeta)^{-1}\,d\zeta.$$

Proof — next time.