

Math 642

Apr. 20, 2020

Last time: Finding the spectral measure.

Today: Fourier transforms.

1. The spectral measure in terms of Green's Functions.

Recall that if $\mu < \lambda$, then we have that $\lim_{\epsilon \downarrow 0} \left[\frac{1}{2\pi i} \int_{\mu}^{\lambda} (G(x, y, t + i\epsilon) - G(x, y, t - i\epsilon)) dt \right]$ sign was wrong in notes

This is sometimes called Stone's formula. The spectral measure is given in terms of a kernel, which

come from the one for $(L - \lambda I)^{-1}$. This is what

$G(x, y, \lambda)$ is. Finally, if the kernel is

"nice", we can bring the limit through the

integral. If we do that, we get

$$\frac{1}{2} (E_{\lambda} + E_{\lambda'}) - \frac{1}{2} (E_{\mu} + E_{\mu'}) =$$

$$\frac{1}{2\pi i} \int_{\mu}^{\lambda} (G(x, y, t) - G(x, y, \bar{t})) dt$$

$$= \frac{1}{2\pi i} \int_{\mu}^{\lambda} (G(x, y, t) - G(x, y, \bar{t})) dt,$$

2. Green's Function for $L - \lambda I$

$$Lu = -u'', \quad D(L) = \{u \in L^2, u'' \in L^2\}$$

Comments, $L = L^*$ $\Rightarrow \sigma(L) \subseteq \mathbb{R}$.

~~Can show~~ Can show that $\langle Lu, u \rangle \geq 0 \quad \forall u \in D(L)$.

This implies that $\sigma(L) \subseteq [0, \infty)$. In fact,

$$\sigma(L) = \sigma_c(L) = [0, \infty).$$

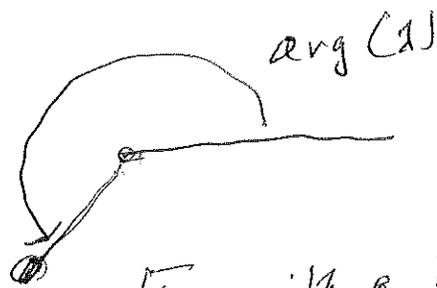
Green's Funct.

$$\lambda \in \rho(L), \quad \left. \begin{array}{l} G''(x, y, \lambda) - \lambda G = \delta(x-y) \\ G(x, y, \lambda) \text{ is in } L^2(\mathbb{R}) \text{ ~~and~~ in } \mathbb{R} \end{array} \right\}$$

Homogeneous solutions $e^{\pm i\sqrt{\lambda}(x-y)}$

Choice of $\sqrt{\lambda}$. Since we want to avoid $\sigma(L)$, we first choose

$$0 < \arg(\lambda) < 2\pi.$$



we then choose $\sqrt{\lambda}$ with a branch cut along the nonnegative real axis:

$$\sqrt{\lambda} = |\lambda|^{1/2} e^{i \frac{1}{2} \arg(\lambda)}$$



Note that with this choice, $\text{Im}(\sqrt{\lambda}) > 0$, because the $\sqrt{\lambda}$ is in the upper half plane.

• Choice of sign

For $x > y$, $G(x, y, \lambda) \in L^2[y, \infty]$ and

For $x < y$, $G(x, y, \lambda) \in L^2(-\infty, y]$.

We have $e^{\pm i\sqrt{\lambda}(x-y)} = e^{\pm \operatorname{Re}(i\sqrt{\lambda})(x-y)}$

~~We have~~ $\operatorname{Re}(i\sqrt{\lambda}) = -\frac{\operatorname{Im}(\sqrt{\lambda})}{\lambda} < 0$.

~~If~~ $\Rightarrow \operatorname{Re}(i\sqrt{\lambda}(x-y)) = -\operatorname{Im}(\sqrt{\lambda})(x-y)$.

For $x > y$, use $+$. For $x < y$, $x-y < 0$,

choose $-$.

$$G(x, y, \lambda) = \begin{cases} Ae^{-i\sqrt{\lambda}(x-y)} & x < y \\ Be^{+i\sqrt{\lambda}(x-y)} & x > y \end{cases}$$

$$\Rightarrow G(x, y, \lambda) = \begin{cases} A e^{+i\sqrt{\lambda}(y-x)}, & x < y \\ B e^{i\sqrt{\lambda}(x-y)}, & x > y. \end{cases}$$

Continuity. $G(y^+, y, \lambda) = G(y^-, y, \lambda)$
 $B = A$

$A = B = \frac{i}{2\sqrt{\lambda}}$

Jump $G'(y^+, y, \lambda) - G'(y^-, y, \lambda) = -1$.

$$\Rightarrow B(+i\sqrt{\lambda}) - (-i\sqrt{\lambda})A = -1 \Rightarrow A = B = \frac{-1}{i\sqrt{\lambda}}$$

$$\Rightarrow G(x, y, \lambda) = \frac{i}{2\sqrt{\lambda}} \begin{cases} e^{-i\sqrt{\lambda}(x-y)}, & x < y \\ e^{i\sqrt{\lambda}(x-y)}, & x > y. \end{cases}$$

$$\Rightarrow G(x, y, \lambda) = \frac{i}{2\sqrt{\lambda}} \begin{cases} e^{i\sqrt{\lambda}(y-x)}, & x < y \\ e^{i\sqrt{\lambda}(x-y)}, & x > y. \end{cases}$$

$$\Rightarrow G(x, y, \lambda) = \frac{i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda}|x-y|}$$

Consequences

(i) Because $\sqrt{\lambda}$ is analytic in $\lambda \notin [0, \infty)$, $G(x, y, \lambda)$ is also analytic there. In addition, by construction, for such λ ,

$$(L - \lambda I)^{-1} f \equiv \int_{-\infty}^{\infty} G(x, y, \lambda) f(y) dy$$

defines a hld. op. \Rightarrow If $\lambda \notin [0, \infty)$, then $\lambda \in \rho(L)$. Also, $\sigma(L) \subseteq [0, \infty)$.

(ii) ~~If $\lambda \in (-\infty, 0)$, then E_λ~~

(iii) If $\mu < 0$, then $E_\mu = 0$. pncii holds because E_μ is constant in any interval

~~in \mathbb{R} where $\mu \in \mathbb{R}$ where the interval is in $\rho(L)$. Thus $E_\mu = \text{const. in } (-\infty, 0)$.~~

But, $s\text{-}\lim_{\mu \rightarrow -\infty} E_\mu = 0$, so $E_\mu = 0$. ~~Also, $E_\mu = 0$~~

(iii) It is easy to show that L has no eigenvectors, so there are no eigenvalues; thus $\sigma_{\text{disc}}(L) = \emptyset$.

One can use this to show that $\mu \leq 0$
 $E_{\lambda^-} = E_{\lambda}$ in all λ . Thus, if $\lambda > 0$, $\mu \leq 0$

$$\frac{E_{\lambda} + E_{\lambda^-}}{2} - \frac{E_{\lambda} E_{\lambda^-}}{2} = E_{\lambda} - 0$$

$$\frac{1}{2\pi i} \int_0^{\lambda} (G(x, y, t^+) - G(x, y, t^-)) dt$$

$$\Rightarrow E_{\lambda} = \int_0^{\lambda} \frac{G(x, y, t^+) - G(x, y, t^-)}{2\pi i} dt$$

3. Spectral measure.

Exercise ~~On~~ Let $t > 0$. Then,

$$\lim_{\varepsilon \downarrow 0} \overline{t \pm i\varepsilon} = \begin{cases} \sqrt{t} \\ -\sqrt{t} \end{cases}$$

$$\Rightarrow G(x, y, t^+) = \frac{i}{2\sqrt{t}} e^{i\sqrt{t}|x-y|}, \quad G(x, y, t^-) = \frac{-i}{2\sqrt{t}} e^{-i\sqrt{t}|x-y|}$$

$$\Rightarrow E_{\lambda} = \frac{1}{2\pi i} \int_0^{\lambda} \frac{i}{2\sqrt{t}} (e^{i\sqrt{t}|x-y|} - e^{-i\sqrt{t}|x-y|}) dt$$

$$\text{Let } u = \sqrt{t}, \quad E_{\lambda} = \frac{1}{2\pi} \int_0^{\sqrt{\lambda}} \omega(\sqrt{u}|x-y|) du$$