Math 642
April 27, 2020



Last time: Properties of the FT + Sampling Thm.
Today: ~~Proof of the sampling thm~~. Proof of Samp. Thm + Asymptotic analysis.

1. Sampling Theorem. $f \in L^2(\mathbb{R})$, $\operatorname{supp}(\hat{f}) \subseteq [-\Omega, \Omega]$,

$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\Omega}\right) \frac{\sin(\Omega x - k\pi)}{\Omega x - k\pi}.$$

Proof: ~~Exam~~ Expand $\hat{f}(\xi)$ in a Fourier series on the interval* $[-\Omega, \Omega]$:

(*) $$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{in\pi x}{\Omega}}, \quad a_n = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{-\frac{in\pi\xi}{\Omega}} d\xi.$$

Note: $f(x) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{+i\xi x} d\xi$, ~~so~~ so

(Inversion thm.)

$$a_n = \frac{\pi}{\Omega} \left( \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{-i\left(\frac{n\pi}{\Omega}\right)\xi} d\xi \right) = \frac{\pi}{\Omega} f\left(-\frac{n\pi}{\Omega}\right).$$

In (*), change $n \to -k$. Then,

$$\hat{f}(\xi) = \sum_{k=-\infty}^{\infty} a_{-k} e^{-\frac{ik\pi x}{\Omega}} = \sum_{k=-\infty}^{\infty} \frac{\pi}{\Omega} f\left(\frac{k\pi}{\Omega}\right) e^{\frac{-ik\pi\xi}{\Omega}},$$

$$f(x) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{+i\xi x} d\xi = \sum_{k=-\infty}^{\infty} \frac{\pi}{\Omega} f\left(\frac{k\pi}{\Omega}\right) \underbrace{\int_{-\Omega}^{\Omega} e^{i\xi\left(x - \frac{k\pi}{\Omega}\right)} d\xi}_{= \frac{\sin(\Omega x - k\pi)}{x - \frac{k\pi}{\Omega}}}.$$

~~Example~~ $\Rightarrow$ $$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\Omega}\right) \frac{\sin(\Omega x - k\pi)}{\Omega x - k\pi}.$$

* Scale the usual formula for the Four. series, ~~i.e., we use $f(x+2\pi) = f(x)$~~ $x \to \frac{\pi x}{\Omega}$, etc.

## 2 Asymptotic Analysis

Basic terms We are going to discuss, mainly, large $x$ behavior of functions.

Big "Oh" notation $f(x) = O(x^{-\alpha})$ as $x \to \infty$ means there is a constant $C$ independent of $x$; $\alpha$ such that $|f(x)| \le C x^{-\alpha}$, for $x$ large.

Little "oh" $f(x) = o(x^{-\alpha})$ means that

$$\lim_{x \to \infty} \frac{f(x)}{x^{\alpha}} = 0 .$$

Examples. $\frac{1}{x^2+1} = O(x^{-2})$ and $\frac{1}{x^2+1} - x^{-2} = o(x^{-2})$.

Asymptotics. $\Gamma(x+1) = \sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x}\left(1 + O(x^{-1})\right)$.

This means that $\frac{\Gamma(x+1)}{\sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x}} = 1 + O(x^{-1})$.

~~Note: $\lim_{x \to \infty} \Gamma(x+1) = +\infty$.~~

Note:

$$\Gamma(x+1) - \sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x} = O\left(x^{x-\frac{1}{2}} e^{-x}\right) \to \infty .$$

We are not saying that the difference goes to 0, only the ratio $-1$.

Relative error vs. abs. error

Avogadro's number $6.02 \times 10^{23}$ ← If we drop the .02, it's still a good relative

Relative error vs. absolute error  Avogadro's number $\approx 6.02 \times 10^{23}$

If we drop $0.02 \times 10^{23}$, it's still a reasonable approximation — in a relative sense!

$$\frac{6.02 \times 10^{23}}{6 \times 10^{23}} \approx 1 + 10^{-3} \approx \frac{1}{3} \approx 1 + 3 \times 10^{-4},$$

Absolute error is $6.02 \times 10^{23} - 6.00 \times 10^{23} = 2 \times 10^{21}$ ← which is huge!!

Asymptotic approximations are generally relative.

Asymptotic Series  It turns out that, in

$$\Gamma(1+x) = \sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right),$$

where equality is taken in an asymptotic sense. The series $1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots$ is divergent. What we mean is that we will get a better approximation if we add terms — but, this will only be better for $x$ taken to be very large.

Another Example  $Ei(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt \overset{t = x(u+1)}{=} e^{-x} \int_0^\infty \frac{e^{-xu}}{1+u}\, du.$

has the asymptotic expansion

$Ei(x) \sim e^{-x} \left( \sum_{k=0}^{\infty} k! (-1)^k x^{-k} \right)$, which is clearly divergent.

3. Watson's Lemma Frequently, we want to find asymptotic expansions for quantities like

$$\int_0^\infty f(t)e^{-xt}, \quad \int_{-a}^\infty f(t)e^{-xt}dt, \text{ etc.}$$

Watson's Lemma Suppose that $f(t) \in L_p([0,\infty))$ and that for $0 \le t \le T$, $f(t) = \sum_{k=1}^{\infty} a_k t^{\frac{k}{n}-1}$, then

(*) $F(x) = \int_0^\infty f(t)e^{-xt}dt \sim \sum_{k=1}^{\infty} a_k x^{-\frac{k}{n}} \Gamma\left(\frac{k}{n}\right)$.

Remark. (1) $f(t)$ doesn't need to have a complete series expansion, only one with a finite number of terms will do.

(2) The expansion that one gets, namely (*), is exactly what you would get if you interchanges (improperly!) the sum and integral:

$$\sum_{k=1}^{\infty} \int_0^\infty f(t)e^{-xt}dt = \sum_{k=1}^{\infty} a_k \int_0^\infty t^{\frac{k}{n}-1}e^{-xt}dt$$

$$\hookrightarrow u = xt, \text{ interpreted} = x^{-k/n}\Gamma\left(\frac{k}{n}\right)$$

So, even though it's wrong the end result is correct.

Proof: (*) $F(x) = \int_0^T f(t)e^{-xt}dt + \int_T^\infty f(t)e^{-xt}dt$.

(1) $\left|\int_T^\infty f(t)e^{-xt}\right| \le \|f\|_{L^p([T,\infty))} \|e^{-xt}\|_{L^q([T,\infty))}$

$\|e^{-xt}\|_{L^q([T,\infty))} = \left(\int_T^\infty e^{-qxt}dt\right)^{1/q} = \left(\frac{e^{-qxT}}{qx}\right)^{1/q} = q^{-1/q}x^{-1/q}e^{-xT}$.

$$\Rightarrow \left| \int_T f(t) e^{-xt} dt \right| \leq \|f\|_{L^p[T,\infty)} \cdot \|e^{-xt}\|_{L^q[T,\infty)}$$

$$\Rightarrow \left| \int_T f(t) e^{-xt} \right| \leq \|f\|_{L^p[T,\infty)} \cdot \frac{1}{q^{1/q}} x^{-1/q} e^{-xT}.$$

$$\Rightarrow \left| \int_T^\infty f(t) e^{-xt} dt \right| = O(x^{-1/q} e^{-xT}).$$

(2) We will need another inequality:

$$\int_T^\infty t^{\alpha-1} e^{-xt} dt \leq C x^{-\alpha} e^{-\frac{xT}{2}} \Gamma(\alpha)$$

$$C \Gamma(\alpha) x^{-\alpha} e^{-\frac{xT}{2}}.$$

Proof: $\int_T^\infty t^{\alpha-1} e^{-xt} dt = \int_T^\infty t^{\alpha-1} e^{-\frac{xt}{2}} \cdot e^{-\frac{xt}{2}}$

$$\leq \left( \int_T^\infty t^{\alpha-1} e^{-\frac{xt}{2}} dt \right) \|e^{-\frac{xt}{2}}\|_{L^\infty[T,\infty)}$$

$$\leq \left( \int_0^\infty t^{\alpha-1} e^{-\frac{xt}{2}} dt \right) \cdot e^{-\frac{xT}{2}}$$

$$= (2/x)^\alpha \Gamma(\alpha) e^{-xT/2},$$

$$\leq 2^\alpha x^{-\alpha} \Gamma(\alpha) e^{-xT/2}.$$

Finish this up next time.