Last time: Properties of the FT + Sampling Theory.


1. Sampling Theorem. \( f \in L^1(\mathbb{R}) \), suppose \( \hat{f}(\xi) \in L^\infty(\mathbb{R}) \).

\[
\hat{f}(\xi) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{\sin(\pi(2\pi - \xi k))}{\pi(2\pi - \xi k)}.
\]

Proof: Expand \( \hat{f}(\xi) \) in a Fourier series, not in the interval \([-\pi, \pi]\):

\[
(\xi) \quad \hat{f}(\xi) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{-i\pi nx}{\pi}} \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{f}(\xi) e^{-i\pi nx} d\xi.
\]

Note: \( f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{f}(\xi) e^{i\pi \xi x} d\xi \) \( \forall \xi \in \mathbb{R} \).

Inversion theorem:

\[
a_n = \frac{1}{\pi} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\xi) e^{-i\pi nx} d\xi \right) = \frac{1}{\pi} + \frac{1}{\pi} - \frac{\pi n}{2}.
\]

In (1), change \( n \to -k \): Then

\[
\hat{f}(\xi) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{\sin(\pi(2\pi - \xi k))}{\pi(2\pi - \xi k)}.
\]

\[
f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \hat{f}(\xi) e^{i\pi \xi x} d\xi = \sum_{k=-\infty}^{\infty} \frac{e^{i\pi \xi k}}{\pi} \hat{f}(k) \frac{\sin(\pi(2\pi - \xi k))}{\pi(2\pi - \xi k)}
\]

\[
\hat{f}(\xi) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) e^{-i\pi \xi x} d\xi = \sum_{k=-\infty}^{\infty} \frac{e^{-i\pi \xi k}}{\pi} \hat{f}(k) \frac{\sin(\pi(2\pi - \xi k))}{\pi(2\pi - \xi k)}
\]

\[
f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{\sin(\pi(2\pi - \xi k))}{\pi(2\pi - \xi k)}
\]

A scale the usual formula for the Fourier series:

\[ f(x) = \hat{f}(\xi) e^{i\pi \xi x} d\xi \]

\[ \hat{f}(\xi) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) e^{-i\pi \xi x} d\xi \]
2. Asymptotic Analysis

Basic terms: We are going to discuss, mainly, large-$x$ behavior of functions.

A Big $\mathcal{O}$ notation $f(x) = \mathcal{O}(x^{-d})$ as $x \to \infty$ means there is a constant $C$ independent of $x$ such that $|f(x)| \leq C x^{-d}$ for $x$ large.

A Little $o$ notation $f(x) = o(x^{-d})$ means that

$$\lim_{x \to \infty} \frac{f(x)}{x^d} = 0.$$ 

Examples. \( \frac{1}{x^2 + 1} = \mathcal{O}(x^{-2}) \) and \( \frac{1}{x^{1/4}} - x^{-2} = o(x^{-2}) \).

Asymptotics. (a) \( \Gamma(x) = \frac{x^{x-1/2} e^{-x}}{\sqrt{2\pi x}} \Gamma(1 + O(x^{-1})) \).

This means that \( \frac{\Gamma(x+1)}{e^{x+1/2} \sqrt{2\pi x}} = 1 + O(x^{-1}) \).

Note: \( \lim_{x \to \infty} \Gamma(x+1) = +\infty \).

Note:
\[ \Gamma(x+1) - x^{x+1/2} e^{-x} = \mathcal{O}(x^{-d} e^{-x}) \to 0, \]

We are not saying that the difference goes to 0, only the ratio $-1$.

Relative error w.r.t. abs. error: $\frac{\text{Avg no. of nuc}}{6.02 \times 10^{23}} \leq 0$. If we drop the $0$, it's still a good relative...
If we drop $6.02 	imes 10^{23}$, it's still a reasonable approximation — in a relative sense:

$$\frac{6.02 \times 10^{23}}{6 \times 10^{23}} = 1 + 10^{-3} \approx 1 + 3 \times 10^{-4}$$

Its absolute error is $6.02 \times 10^{23} - 6.02 \times 10^{23} = 2 \times 10^{21}$ — which is huge!!

Asymptotic approximations are generally relative.

Asymptotic sense: It turns out that for

$$V(x) = \sqrt{2\pi} x^{2.5 + \epsilon} e^{-x} \left(1 + \frac{5}{12x} + \frac{1}{288x^2} + \cdots\right)$$

where equality is taken in an asymptotic sense. The series $1 + \frac{5}{12x} + \frac{1}{288x^2} + \cdots$ diverges. What we mean is that we will get a good approximation if we add terms — but, this will only be better for $x$ taken to be very large.

Another example $E_1(x) = \int_x^\infty \frac{e^{-u}}{u} \, du$ has the asymptotic expansion

$$E_1(x) \approx e^{-x} \left( \sum_{k=0}^{\infty} \frac{k!}{(-1)^k x^k} \right)$$

which is clearly divergent.
3. **Watson's Lemma** frequently, we want to find asymptotic expansions for quantities like

\[
\int_0^\infty f(t) e^{-kt} dt, \int_0^\infty e^{-kt} dt, \text{ etc.}
\]

**Watson's Lemma** Suppose that \( f(t) \in L_p((0,\infty)) \) and that for \( 0 < t < \infty \), \( f(t) = \sum_{k=1}^\infty a_k t^{-\frac{k}{p} - 1} \), then

\[
F(t) = \int_0^\infty f(t) e^{-kt} dt = \sum_{k=1}^\infty a_k \int_0^\infty e^{-kt} t^{-\frac{k}{p}} dt = \sum_{k=1}^\infty a_k \frac{1}{\Gamma\left(\frac{k}{p}\right)}.
\]

**Remark.**

1. \( f(t) \) doesn't need to have a complete series expansion, only one with a finite number of terms will do.

2. The expansion that one gets, namely \( a_k \), is just exactly what we would get if you interchanged \((\text{improperly!})\) the sum and integral:

\[
\int_0^\infty f(t) e^{-kt} dt = \sum_{k=1}^\infty a_k \int_0^\infty e^{-kt} t^{-\frac{k}{p}} dt
\]

\[
F(t) = \sum_{k=1}^\infty a_k \frac{1}{\Gamma\left(\frac{k}{p}\right)} = e^{-\frac{t}{p}}\Gamma\left(\frac{1}{p}\right),
\]

so even though it's wrong, \( W \) in a result is correct.

**Proof:**

\[
F(t) = \int_0^t f(t) e^{-kt} dt + \int_t^\infty f(t) e^{-kt} dt.
\]

\[
|\int_t^\infty f(t) e^{-kt} dt| \leq \|f\|_L^p(0,\infty) \int_t^\infty e^{-kt} dt = \frac{1}{k} e^{-kt} \bigg|_t^\infty = \frac{1}{k} e^{-kt},
\]

\[
\|f\|_L^p(0,\infty) \leq \frac{1}{k} e^{-kt} \int_t^\infty e^{-kt} dt = \frac{1}{k} e^{-kt} \bigg|_t^\infty = \frac{1}{k} e^{-kt}.
\]
\[
\int_0^\infty |f(t) e^{-\lambda t}| dt \leq \int_0^\infty |\xi f'(t) e^{-\lambda t}| dt \\
= \int_0^\infty |f(t) e^{-\lambda t}| dt \\
= \int_0^\infty f(t) e^{-\lambda t} dt = O(e^{-\lambda T}).
\]

(2) We will need another inequality:
\[
\int_0^\infty x e^{-\lambda t} dt \leq \frac{e^{-\lambda t}}{t^2}.
\]

**Proof:**
\[
\int_0^\infty x e^{-\lambda t} dt = \int_0^\infty x e^{-\frac{x}{2} - \frac{x}{t}} dt \\
\leq \left( \int_0^\infty x e^{-\frac{x}{2}} dt \right) \frac{e^{-\frac{x}{2}}}{t^2} \\
\leq \left( \int_0^\infty x e^{-\frac{x}{2}} dt \right) e^{-\frac{x}{2}} \\
= \left( \frac{2}{\lambda^2} \right) \frac{e^{-\frac{x}{2}}}{t^2} \\
\leq 2 e^{-\lambda T/2}.
\]

Finish this up next time.