

Math 642  
AM. 28, 2020

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Last term: Started asymptotics, Watson's Lemma

Today: Watson's lemma, Stirling's formula.

1. Watson's Lemma. Let  $F(x) = \int_0^\infty f(t) e^{-xt} dt$ . If  $f(t) = \sum_{n=1}^{\infty} a_n t^{\frac{k}{n}-1}$  uniformly on  $[t, T]$ , and is in  $L^p([t, T])$ , then

$$F(x) \approx \sum_{n=1}^{\infty} a_n V\left(\frac{k}{n}\right) x^{-\frac{k}{n}},$$

Proof: Last time  $\left| \int_T^\infty e^{-xt} f(t) dt \right| \leq \|f\|_{L^p([T, \infty))} e^{-kT}$ ,

~~$$\text{Also, } \int_T^\infty t^{\frac{k}{n}-1} e^{-xt} dt \leq \int_T^\infty$$~~

$$\begin{aligned} \text{Also, } \int_T^\infty t^{\frac{k}{n}-1} e^{-xt} dt &= \int_T^\infty t^{\frac{k}{n}-1} e^{-\frac{x}{2}} e^{-\frac{xt}{2}} dt \\ &\leq \left( \int_T^\infty t^{\frac{k}{n}-1} e^{-\frac{xt}{2}} dt \right) \|e^{-\frac{xt}{2}}\|_\infty \\ &= e^{-\frac{xt}{2}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_T^\infty t^{\frac{k}{n}-1} \int_T^\infty t^{\frac{k}{n}-1} e^{-xt} dt &\leq e^{-\frac{xt}{2}} \int_T^\infty t^{\frac{k}{n}-1} e^{-\frac{xt}{2}} dt \\ &\leq e^{-\frac{xt}{2}} \int_0^\infty t^{\frac{k}{n}-1} e^{-\frac{xt}{2}} dt \\ \Rightarrow \int_0^\infty t^{\frac{k}{n}-1} e^{-xt} &\leq \left(\frac{\pi}{x}\right)^{-\frac{k}{n}} e^{-\frac{xt}{2}} L\left(\frac{k}{n}\right), \end{aligned}$$

$$\text{So, } F(x) = \int_0^T e^{-xt} f(t) dt + \int_T^\infty e^{-xt} f(t) dt$$

$$F(x) = \sum_{n=1}^{\infty} a_n \left( \int_0^T t^{\frac{k}{n}-1} e^{-xt} dt \right) + O(e^{-xt})$$

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$$\Rightarrow F(x) = \sum_{k=0}^{\infty} a_k \left( \int_0^x t^{\frac{k}{n}-1} e^{-xt} dt - \int_x^{\infty} t^{\frac{k}{n}-1} e^{-xt} dt \right) + O(e^{-xT}),$$

$$\text{Since } \int_0^x t^{\frac{k}{n}-1} e^{-xt} dt = I^*(\frac{k}{n}) x^{-\frac{k}{n}} \text{ and}$$

$$\int_x^{\infty} t^{\frac{k}{n}-1} e^{-xt} dt \leq (\frac{x}{e})^{-\frac{k}{n}} e^{-\frac{kT}{2}} \cdot I^*(\frac{k}{n})$$

$$\Rightarrow \int_x^{\infty} t^{\frac{k}{n}-1} e^{-xt} dt = O(x^{-\frac{k}{n}} \cdot e^{-xT/2})$$

we have  $F(x) \approx \sum_{k=0}^{\infty} I^*(\frac{k}{n}) a_k x^{-\frac{k}{n}} + O(e^{-\frac{T}{2}x})$ ,  
which completes the proof

Other version.

$$F(x) = \int_0^{\infty} f(t) e^{-xt} dt, \quad f(t) = \sum_{k=0}^{\infty} a_k t^{\frac{k}{n}}, \quad |t| \leq T,$$

$$F(x) \approx \frac{1}{n} \sum_{k=1}^{\infty} a_{k-1} I^*(\frac{k}{n}) x^{-\frac{k-1}{n}}$$

$$F(x) \approx \int_{-x}^{\infty} f(t) e^{-xt} dt, \quad f(t) = \sum_{k=0}^{\infty} a_k t^{\frac{k}{n}}, \quad |t| \leq T,$$

$$F(x) \approx \sum_{k=0}^{\infty} a_{2k} I^*(\frac{k+\frac{1}{2}}{n}) x^{-\frac{k+\frac{1}{2}}{n}}$$

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Stirling's

Stirling's Formulae

$$\Gamma(x+1) \sim \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} \left(1 + O\left(\frac{1}{x}\right)\right), \quad x \rightarrow \infty.$$

Proof:  $\Gamma(x+1) = \int_0^\infty t^{x+1} e^{-t} dt = \int_0^\infty (vu)^x e^{-vu} u^x du.$

$$\Rightarrow \Gamma(x+1) = x^{x+1} \int_0^\infty u^x e^{-vu} du$$

$u = \frac{t}{x}, \quad du = \frac{1}{x} dt$

With  $u^x e^{-vu} = e^{x \ln u - vu} = e^{x \ln u - v(u - \ln u)}$

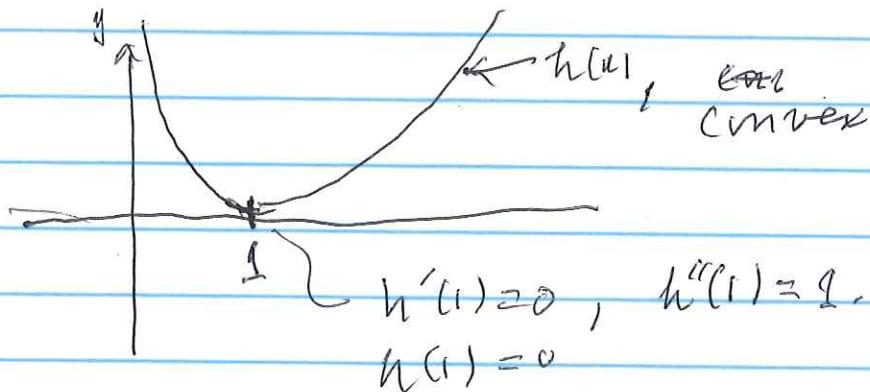
$$= e^{-v(u - \ln u)}.$$

For convenience, add and subtract  $\frac{1}{2}$ :

$$u^x e^{-vu} = e^{-v} \cdot e^{-v(u - 1 - \ln u)},$$

Let  $h(u) := u - 1 - \ln u$ . Then

$$\Gamma(x+1) = x^{x+1} e^{-x} \int_0^\infty e^{-v h(u)} du.$$

Graph of  $h(u)$ 

Thus, near  $u=1$ ,

$$h(u) = h(1) + h'(1)(u-1) + \frac{1}{2} h''(1)(u-1)^2 + \dots$$

$$\therefore h(u) \leq \frac{1}{2} h''(1)(u-1)^2 + \text{higher order terms.}$$

Note that  $h(u)$  is essentially quadratic near 0  
Note that we can't can

Define  $v(u) = \sqrt{h(u)} \operatorname{sgn}(u-1)$ . For  $u \neq 1$ ,  
 This a differentiable fnct. How does it behave near 1?

$$h(u) = \frac{1}{2} [(u-1)^2] [1 + a_1(u-1) + a_2(u-1)^2 \dots]$$

Taylor series for  $h(u)$ .

$$\Rightarrow \pm \sqrt{h(u)} = \frac{1}{2} (u-1) \underbrace{\left( 1 + a_1(u-1) \dots \right)}_{\text{analytic}}^{Y_2}$$

By choosing the sign to be + if  $u > 1$ , and - if  $u < 1$ ,  
 we see that near  $u=1$ ,

$$(*) \quad v(u) = \frac{1}{2} (u-1) \left( 1 + a_1(u-1) \dots \right)^{Y_2}.$$

That is,  ~~$\sqrt{h(u)}$~~  is not only differentiable, but analytic  
 near  $u=1$ . Change variables: let solve  
 for  ~~$\sqrt{h(u)}$~~   $u = u(v)$ , then  $du = u'(v) dv$   
 and

$$h(u) = v^{k+1} e^{-v} \int_{-\infty}^v e^{-x} v^{k+1} \frac{du}{dv} dx = h(u)$$

By (\*), near  $v=0$ ,  $v \approx \frac{1}{2}(u-1) \dots$ , so  
 we can invert & find  $u=u(v)$ . ~~This process~~ In  
 fact,  $u=u(v)$  is analytic near  $v=0$ . ~~near~~

$$\Rightarrow u = 1 + a_1 v + a_2 v^2 + \dots$$

~~$\Rightarrow \frac{du}{dv} \Big|_{v=0} = \frac{1}{a_1} \Big|_{v=0} = \frac{1}{a_1} = \frac{1}{\sqrt{2}}$~~

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$$\rightarrow \frac{du}{dv} = a_1, \text{ but, } \left. \frac{du}{dv} \right|_{v=0} = \left( \left. \frac{du}{dv} \right|_{u=1} \right)^{-1}$$

so  $\left. \frac{du}{dv} \right|_{v=0} = \left( \frac{1}{\sqrt{2}} \right) = \sqrt{2}$ .

One can get all terms in the series this way. We will just do the first term. Near  $v=0$ , so

$$\frac{du}{dv} = \sqrt{2}, \quad \text{so}$$

$$\frac{du}{dv} = \sqrt{2} + 2a_1 v + \dots$$

$$\begin{aligned} \Rightarrow P(v+1) &= v^{k+\frac{1}{2}} e^{-x} \left( \sqrt{2} \cdot \left( \int_{-\infty}^{\infty} e^{-x} \frac{v^2}{dv} dv \right) + \dots \right), \quad x \rightarrow \infty \\ &= x^{k+\frac{1}{2}} e^{-x} \left( \sqrt{2\pi} x^{-(1/2)} + \dots \right) \quad \text{as } v \rightarrow \infty \\ &= x^{k+\frac{1}{2}} e^{-x} \sqrt{\pi} (1 + \text{higher order terms}). \end{aligned}$$

$$\therefore P(v+1) \approx x^{k+\frac{1}{2}} e^{-x} \sqrt{\pi} (1 + O(x^{-1})).$$