

Math 641
April 3, 2020

Last time: Operator-analytic filters, resolvent sets, resolvents, \mathbb{I}^{\perp} resolvent identity.

Today: Results in interpolation and projection.

1. Interpolation. Suppose that ~~$F: \mathbb{C} \rightarrow \mathbb{C}$~~ $F: [a, b] \rightarrow \mathbb{C}$
s.t. $F(t)$ is continuous at t_* i.e.

$$\lim_{\delta \rightarrow 0} \|F(t+\delta) - F(t)\| = 0,$$

The ~~int~~ integral $\int_a^b F(t) dt$ is defined in terms of a Riemann sum:

$$\int_a^b F(t) dt = \lim_{\|P\| \rightarrow 0} \sum_{j=0}^{n-1} F(t_j^*) (t_{j+1} - t_j),$$

where $P = \{t_0 < t_1 < \dots < t_n\}$, and $t_j^* \in [t_j, t_{j+1}]$. The limit is in the operator norm.

2. Contour integration. As usual,

$$\int_C F(z) dz = \int_a^b F(z(t)) z'(t) dt,$$

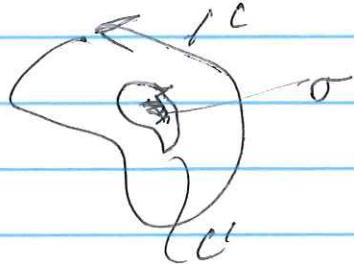
when $z(t)$ parameterizes C , and $F: \omega \rightarrow \mathcal{B}(\mathbb{H})$,

Here, ω is a domain in \mathbb{C} open, connected. These theorems hold: ~~Cauchy's Thm~~, Cauchy's Thm, Cauchy's Thm for multiply connected domains, etc.

3. Integrals involving resolvents.

Let L be an operator defined on a closed and densely defined. Suppose that σ is an isolated part of the spectrum of L . We then have this result:

Lemma Consider the situation shown in the figure below. Then,



$$\int_C (L - \lambda I)^{-1} d\lambda = \int_{C'} (I - \lambda L)^{-1} d\lambda.$$

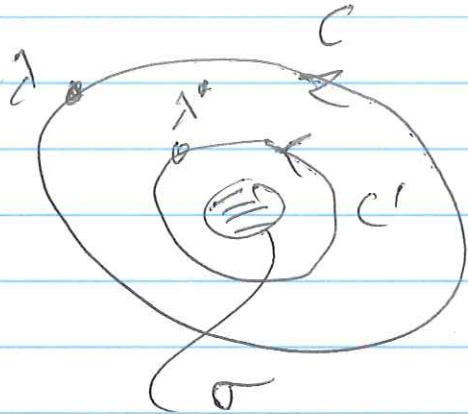
Proof: Standard argument from complex analysis.

Proposition. Let C and σ be as above. Then,

$$P_C = -\frac{1}{2\pi i} \oint_C R_\lambda(L) d\lambda$$

is a projection. Moreover, P_C is ~~independent~~ independent of the choice of C — i.e. it depends only on σ .

P₀: Again, consider the figure below:



First of all by the lemma, P_C really and only depends on σ , by the lemma. So, we will write P_σ for P_C . Then,

$$P_\sigma^2 = \left(\frac{1}{2\pi i} \int_C (L - \lambda I)^{-1} d\lambda \right) \left(-\frac{1}{2\pi i} \int_{C'} (L - \lambda' I)^{-1} d\lambda' \right)$$

We may write P_σ^2 as

$$P_\sigma^2 = \frac{1}{4\pi^2} \int_C \int_{C'} (L - \lambda I)^{-1} (L - \lambda' I)^{-1} d\lambda d\lambda'$$

By the ~~1st resolvent~~ resolvent identity, we have

$$R_\lambda(L) - R_{\lambda'}(L) = (\lambda - \lambda') \underset{\lambda}{\underset{\lambda'}{\int}} R_\lambda(L) R(\lambda).$$

$$\Rightarrow P_\sigma^2 = \frac{1}{4\pi^2} \int_C \int_{C'} \left(\frac{R_\lambda(L)}{\lambda - \lambda'} - \frac{R_{\lambda'}(L)}{\lambda - \lambda'} \right) d\lambda d\lambda'$$

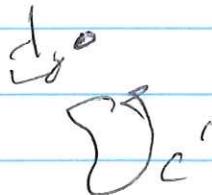
$$P_0^2 = \frac{1}{4\pi^2} \left\{ \int_C R_{\lambda}(z) \left(\int_{C'} \frac{d\lambda'}{\lambda - \lambda'} \right) d\lambda \right\}$$

$$= \int_{C'} R_{\lambda'}(z) \left(\int_C \frac{d\lambda}{\lambda - \lambda'} \right) d\lambda' \}.$$

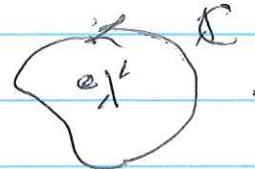
~~By the Consider~~ $\int_{C'} \frac{d\lambda'}{\lambda - \lambda'}$ since λ' is a point

~~inside in interior~~ ~~outside~~ the curve C'

we have $\int_C \frac{d\lambda'}{\lambda - \lambda'} = 0$



Next, consider $\int_C \frac{d\lambda}{\lambda - \lambda'}$.



Since λ' is inside of C ,

$$\int_C \frac{d\lambda}{\lambda - \lambda'} = 2\pi i$$

$$\Rightarrow P_0^2 = \frac{1}{4\pi^2} \left\{ \int_C R_{\lambda}(z) \cdot 0 - 2\pi i \int_C R_{\lambda}(z) d\lambda' \right\}$$

$$P_0^2 = -\frac{1}{2\pi i} \int_{C'} R_{\lambda'}(z) d\lambda' = P_0^1,$$

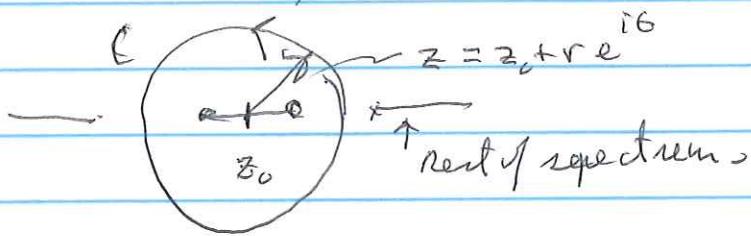
$\therefore P_0$ is a projection.

4. Self-adjoint Operator, Later we will show that the spectrum of a self-adjoint operator is real - i.e., $L = \mathbb{C} \Rightarrow \sigma(L) \subseteq \mathbb{R}$.

[Proof.] Suppose that σ is a line segment disjoint from the rest of $\sigma(L)$.

Since σ is closed, then here must \downarrow be a gap.

Consider the contour,



Let $P_0 = -\frac{1}{2\pi i} \oint_C (L-\lambda I)^{-1} d\lambda$. (Recall the integral is independent of C.)

Prop. $P_0 = P_0^*$, so it is an orthogonal projector

$$\text{Proof: } P_0^* = -\frac{1}{2\pi i} \oint_C (L-\bar{\lambda} I)^{-1} d\bar{\lambda}$$

$$= -\frac{1}{2\pi i} \left(\int_C (L-\bar{\lambda} I)^{-1} d\bar{\lambda} \right)^* \begin{cases} L = L^* \\ (L-\bar{\lambda} I)^* = (L-\bar{\lambda} I) \end{cases}$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} (L - re^{-i\theta})^{-1} e^{i\theta} dt$$

$\stackrel{?}{=} \int_{-\infty}^{\infty}$

$$= \frac{1}{2\pi i} \int_{\Gamma}^{\infty} (z - re^{i\theta})^{-1} (-re^{i\theta}) \, dz$$

$\theta = -45^\circ$

$$= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} (z - re^{i\theta})^{-1} re^{i\theta} \, dz$$

$$= -\frac{1}{2\pi i} \oint_C (z - 1)^{-1} dz,$$

$$= P_0$$

$$\therefore \boxed{P_0^* = P_0}$$

For suppose L is a bdd. operator, Then if
 C encircles

Ex. Suppose $\sigma = \sigma_1 \cup \sigma_2$ $\overset{\sigma}{\overbrace{\sigma_1 \quad \sigma_2}}$,

Then $P_0 = P_{\sigma_1} + P_{\sigma_2}$, $P_{\sigma_1} P_{\sigma_2} = P_{\sigma_2} P_{\sigma_1} = 0$.

Now. $P_0^2 = P_{\sigma_1}^2 + \underbrace{P_{\sigma_1} P_{\sigma_2}}_{=} + P_{\sigma_2} P_{\sigma_1} + P_2^2 = P_{\sigma_1} + P_2 + 2P_{\sigma_1} P_{\sigma_2} = P_{\sigma_1} + P_{\sigma_2}$

$= P_{\sigma_1} P_{\sigma_2} = P_{\sigma_2} P_{\sigma_1} = 0$ $\boxed{= P_{\sigma_1} P_{\sigma_2}}$