

Math 642

October 2020
4/6/2020

(1)

Last time: 1) If $\sigma \subseteq \sigma(L)$, where σ is a compact subset of $\sigma(L)$ that is isolated, then if C is as shown.



$$P_{\sigma} = -\frac{1}{2\pi i} \int_C (L-\lambda)^{-1} d\lambda.$$

2) Assuming that $L = L^*$ and that $\sigma(L) \subseteq \mathbb{R}$, we have that $P_{\sigma} = P_{\sigma}^*$.

3) If $L = L^*$ and $\sigma = \sigma_1 \cup \sigma_2$, $\sigma_1 \cap \sigma_2 = \emptyset$,

$$\sigma_1 \cup \sigma_2, \text{ then } P_{\sigma_1} + P_{\sigma_2} \text{ is } P_{\sigma_1 \cup \sigma_2}.$$

Today: 1) If $L = L^*$, then $\sigma(L) \subseteq \mathbb{R}$

1. The spectrum of a self adjoint operator is real.

Suppose $A = a + ib$, $b \neq 0$. Then, we have

$$\| (L - a - ib) f \| ^2 = \| L f - (a - ib) f \| ^2$$

$$= \| L f \| ^2 - \langle L f, (a + ib) f \rangle - \langle (a + ib) f, L f \rangle$$

$$+ \| (a + ib) f \| ^2$$

$$< \| L f \| ^2 - \cancel{\langle L f, (a + ib) f \rangle} - \cancel{\langle (a + ib) f, L f \rangle}$$

$$- (a - ib) \langle f, L f \rangle + \| aib \| ^2 \| f \| ^2$$

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2. The spectrum of L is real, $\sigma(L) \subseteq \text{IR}$

Suppose that $\lambda = a+ib$. Then,

$$\|(\lambda - L - i\eta)^{-1}f\|^2 = \|(\lambda - L)^{-1}f - i\eta^{-1}f\|^2$$

$$= \|(\lambda - L)^{-1}f\|^2 - ib \langle (\lambda - L)^{-1}f, f \rangle + ib \underbrace{\langle f, (\lambda - L)^{-1}f \rangle}_{= \langle (\lambda - L)f, f \rangle}$$

~~$= \|(\lambda - L)^{-1}f\|^2 - ib \langle (\lambda - L)^{-1}f, f \rangle + ib \langle (\lambda - L)f, f \rangle$~~

$$\geq \|(\lambda - L)^{-1}f\|^2 - ib \underbrace{\langle (\lambda - L)^{-1}f, f \rangle}_0 + ib \underbrace{\langle (\lambda - L)f, f \rangle}_0$$

$$+ \|ibf\|^2$$

$$= \|(\lambda - L)^{-1}f\|^2 + b^2 \|f\|^2 \geq b^2 \|f\|^2,$$

~~(and, let $g \in \mathcal{H}$)~~

This implies that ~~if~~ $\lambda = a+ib$, $b \neq 0$, can't be an eigenvalue of L . (We already ~~know this~~ know this!) ~~Also,~~

can't

This implies $\lambda = a+ib$, $b \neq 0$, ~~mean that~~ it can't be an eigenvalue of L . If it were an e-val., then $(L - \lambda I)^{-1}f = 0 \Rightarrow b^2 \|f\|^2 = 0 \Rightarrow \|f\|^2 = 0$. Thus,

$$\lambda \in \text{IR}.$$

We also know that $\sigma_r = \emptyset$. This

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leaves us with ~~show~~ ~~to show~~ showing that
 σ_c — the continuous spectrum of L — is real. To see this, let $\lambda = \alpha + i\beta \in \sigma_c$. Then, by definition, ~~λ~~

$$\text{Range}(L - \lambda I) = \mathcal{H},$$

but ~~λ~~ $\notin \text{Range}(L - \lambda I) \neq \mathcal{H}$.
 Let $g \in \mathcal{H}$ be ~~arbitrary~~ arbitrary.
 So since the range of $L - \lambda I$ is ~~some~~ dense,
 there exists a ~~seqn~~ sequence in ~~$L - \lambda I$~~ $D(L)$ such
 that $(L - \lambda I)f_n = g_n$, $\{f_n\}$ and $g_n \rightarrow g$.

(i) f_n is Cauchy and therefore converges.

$$\|g_n - g_m\|^2 = \| (L - \lambda I)(f_n - f_m) \| \geq b^2 \|f_n - f_m\|^2.$$

Since $g_n \rightarrow g$, $\{g_n\}$ is Cauchy and $\|g_n - g_m\| \rightarrow 0$,

thus, $b^2 \|f_n - f_m\|^2 \rightarrow 0$ and we have that
 $\{f_n\}$ is Cauchy. It follows that there
 exists an ~~$f \in \mathcal{H}$~~ $f \in \mathcal{H}$ s.t. $f = \lim_{n \rightarrow \infty} f_n$.

(ii) $(L - \lambda I)f = g \Rightarrow \mathcal{H} = \text{Range}(L - \lambda I)$.

~~Because L~~ Because L Because $L - \lambda I$ is closed,
 and because $(L - \lambda I)f_n = g_n$ are such that $f_n \rightarrow f$
 and $g_n \rightarrow g$, we have $f \in D(L)$ and $g = (L - \lambda I)f$,

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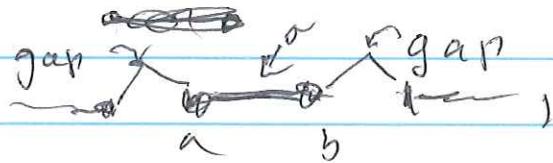
~~We have that~~ It follows that the range of $L - \lambda I$ is \mathcal{H} ; that is,

$$\text{Range}(L - \lambda I) = \mathcal{H}.$$

(iii) since $\lambda \notin \sigma_d$, $\sigma_r = \emptyset$, ~~the~~ $\lambda \notin \sigma_c$, $\lambda \notin \sigma(L)$, we have that $\lambda = \alpha + i\beta \in \rho(L)$, $\beta \neq 0$.

Consequently, $\sigma(L) \subseteq \mathbb{R}$.

This establishes our earlier result: $\sigma = [\alpha, \beta] \subset \sigma(\mathcal{H})$, ~~then~~ and σ is isolated in $\sigma(L)$.



~~We have that~~ $P_{\sigma} \geq P_{\sigma}^*$ is an orthogonal projection. Also, if $\sigma = \sigma_1 \cup \sigma_2$,

$$\sigma \supseteq \sigma_1 \cup \sigma_2$$

then $P_{\sigma} = P_{\sigma_1} + P_{\sigma_2}$. In addition,

$$P_{\sigma_1} P_{\sigma_2} = 0.$$

~~To see this, $P_{\sigma}^2 = P_{\sigma_1}^2 + P_{\sigma_2}^2 + P_{\sigma_1} P_{\sigma_2} + P_{\sigma_2} P_{\sigma_1}$~~

~~$$P_{\sigma} = P_{\sigma_1} + P_{\sigma_2} \Rightarrow P_{\sigma_1} P_{\sigma_2} + P_{\sigma_2} P_{\sigma_1} = 0.$$~~

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$$\text{Because } P_{\sigma_1} = -\frac{1}{2\pi i} \int_{C_1} (L-\lambda_i)^{-1} d\lambda_i \text{ and } P_{\sigma_2} = -\frac{1}{2\pi i} \int_{C_2} (L-\lambda_j)^{-1} d\lambda_j$$

$$\text{and } (L-\lambda_i)^{-1} (L-\lambda_j)^{-1} = (L-\lambda_j) (L-\lambda_i)^{-1}, \text{ we}$$

$$\text{have } P_{\sigma_1} P_{\sigma_2} = P_{\sigma_2} P_{\sigma_1} \Rightarrow \text{we also have}$$

$$P_{\sigma}^2 = P_{\sigma} \Rightarrow (P_{\sigma_1} + P_{\sigma_2})^2 = \underbrace{P_{\sigma_1}^2 + P_{\sigma_2}^2}_{P_{\sigma_1} + P_{\sigma_2}} + P_{\sigma_1} P_{\sigma_2} + P_{\sigma_2} P_{\sigma_1}$$

$$\Rightarrow P_{\sigma_1} + P_{\sigma_2} = P_{\sigma_1} + P_{\sigma_2} + \underbrace{P_{\sigma_1} P_{\sigma_2} + P_{\sigma_2} P_{\sigma_1}}_{=0}$$

$$\Rightarrow 2P_{\sigma_1} P_{\sigma_2} = 2P_{\sigma_2} P_{\sigma_1} = 0$$

$$\therefore P_{\sigma_1} P_{\sigma_2} = P_{\sigma_2} P_{\sigma_1} = 0$$

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Example: $L = -\Delta^2$, $D(L) = \{u \in L^2 : u'' \in L \text{ and } u(0) = u(1)\} = U\}$

Resolvent kernel - (Green's func.)

$$\left. \begin{array}{l} (\text{1-2}) \text{ i.e. } f(x) \\ -\Delta^2 G(x-y) \end{array} \right\} \rightarrow G'' - \Delta G = \delta(x-y) \\ G(0,y) = 0, G(1,y) = 0$$

Standard Standard solution: Take homogeneous solutions for $x \neq y$: At $x < y$, $u_1(x) = \sin(\sqrt{\lambda}x)$ at $x > y$, $u_2(x) = \sin(\sqrt{\lambda}(1-x))$. By (4-8) in the ~~text~~, tent

$$G(x,y,\lambda) = \begin{cases} \frac{\sin(\sqrt{\lambda}(1-y)) \sin(\sqrt{\lambda}x)}{w}, & x \leq y \\ \frac{\sin(\sqrt{\lambda}y) \sin(\sqrt{\lambda}(1-x))}{w}, & x \geq y. \end{cases}$$

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} \sin(\sqrt{\lambda}y) & -\sin(\sqrt{\lambda}(1-y)) \\ \sqrt{\lambda} \sin(\sqrt{\lambda}y) & -\sqrt{\lambda} \sin(\sqrt{\lambda}(1-y)) \end{vmatrix}$$

$$\stackrel{\text{This}}{=} -\sqrt{\lambda} \left(\sin(\sqrt{\lambda}y) \cos(\sqrt{\lambda}(1-y)) + \sin(\sqrt{\lambda}(1-y)) \cos(\sqrt{\lambda}y) \right)$$

$$= -\sqrt{\lambda} \sin(\sqrt{\lambda})$$

$$\therefore G(x,y,\lambda) = \frac{1}{-\sqrt{\lambda} \sin(\sqrt{\lambda})} \begin{cases} \sin(\sqrt{\lambda}(1-y)) \sin(\sqrt{\lambda}x), & x \leq y \\ \sin(\sqrt{\lambda}y) \sin(\sqrt{\lambda}(1-x)), & x \geq y \end{cases}$$

Resolvent op. $(L - \lambda I)^{-1} f = \int_0^1 G(x,y,\lambda) f(y) dy,$