Last line: 1) If \( \sigma \subseteq \sigma_c \), where \( \sigma \) is a compact
subset of \( \sigma_c \) that is violated, then if \( C \) is
as shown,

\[
P = -\frac{1}{2\pi i} \oint_C (z-c)^{-1} dz.
\]

2) Assuming that \( L = L^* \) and that \( \sigma_c \subseteq \sigma \),
we have that \( P = P_y \).

3) If \( L = L^* \) and \( \sigma = \sigma_1 \cup \sigma_2 \), \( \sigma_1 \cap \sigma_2 = \emptyset \),
then \( P + P = \mathbb{1}_P \).

Today: 1) If \( L = L^* \), then \( \sigma_c(L) \subseteq \sigma \).

1. The spectrum of a self-adjoint operator is real.

Suppose \( A = a + i b \), \( b \in \mathbb{C} \). Then, we have

\[
\| (L - a - ib) f \|^2 = \| L f - (a + ib) f \|^2
\]

\[
= \| f \|^2 - \langle L f, (a + ib) f \rangle + \langle (a + ib) f, L f \rangle
\]

\[
= \| f \|^2 - \langle L f, (a + ib) f \rangle - \langle (a + ib) f, L f \rangle
\]

\[
= \| f \|^2 - (a + ib) L f \langle f, f \rangle - \langle (a + ib) f, L f \rangle
\]

\[
= \| f \|^2 - \langle (a + ib) f, L f \rangle - \langle (a - ib) f, L f \rangle - |a + ib|^2 \| f \|^2
\]
1. The spectrum of \( L \) is real, \( \sigma(L) \subseteq \mathbb{R} \)

Suppose that \( A = a + ib \), then,

\[
\| (L - a - ib) f \|^2 = \| (L - a) f - ib f \|^2
\]

\[
= \| (L - a) f \|^2 - ib \langle (L - a) f, f \rangle + ib \langle f, (L - a) f \rangle
\]

\[
+ \| ib f \|^2
\]

\[
= \| (L - a) f \|^2 - ib \langle (L - a) f, f \rangle + ib \langle f, (L - a) f \rangle
\]

\[
+ \| ib f \|^2
\]

\[
= \| (L - a) f \|^2 + b^2 \| f \|^2 \geq b^2 \| f \|^2
\]

\[\text{Must let } \gamma \neq 0\]

This implies that \( \| (L - a) f \|^2 = 0 \), \( \gamma \neq 0 \),

\[\text{can't be an eigenvalue of } L \text{. (We already know this!)} \]

Also,

\[\text{can't}
\]

This implies \( A = a + ib \), \( \gamma \neq 0 \) cannot be an eigenvalue of \( L \). If it were an eigenvalue, then

\[(L - a I) f = 0 \implies b^2 \| f \|^2 = 0 \implies \| f \|^2 = 0 \implies f = 0 \]  \( \text{This is not possible.} \)

\[\| f \|^2 \leq 1 \]

\[\text{We also know that } \sigma_\mathbb{R} = \mathbb{R}. \text{ This}\]
leaves us with showing that 
$ \sigma_c = \text{the continuous spectrum of } \lambda_c$ is real.

To see this, let $\lambda = \alpha + i \beta \in \mathbb{C}$.

Then, by definition,

$$\text{Range}(\lambda - I) = \mathbb{H},$$

but

$$\text{Range}(\lambda - I) \neq \mathbb{H}.$$  

Let $g \in \mathbb{H}$ be arbitrary.

So since the range of $\lambda - I$ is dense, there exists a sequence in $\mathbb{H}$, $(\gamma_n)$, such that $(\lambda - I)\gamma_n = g_\infty \gamma_n$ and $g_n \to g$.  

(i) $f_n$ is Cauchy and therefore convergent.

$$\| f_n - f_m \|^2 = \| (\lambda - I)(f_n - f_m) \|^2 \geq \beta^2 \| f_n - f_m \|^2.$$  

Since $g_n \to g$, $(g_n)$ is Cauchy and $\| g_n - g_m \| \to 0$.

Thus, $\beta^2 \| f_n - f_m \|^2 \to 0$ and we have that $(f_n)$ is Cauchy. It follows that there exists an $f \in \mathcal{H}$ such that $f = \lim_{n \to \infty} f_n$.

(ii) $(\lambda - I)\gamma = g \iff g \in \text{Range}(\lambda - I)$.

Because $\mathbb{C}$ is because $\lambda - I$ is closed, and because $(\lambda - I)\gamma_n = g_n$ are such that $\gamma_n \to \gamma$ and $g_n \to g$, we have $f \in \mathcal{H}$ and $g \in \mathcal{H}$.  


We have that it follows that the range of \( L - 2I \) is \( \mathfrak{F} \); that is,

\[
\text{Range} (L - 2I) = \mathfrak{F}.
\]

Thus, since \( a \in \mathfrak{F} \), \( a = f \), \( \mathfrak{F} + \mathfrak{F} \), \( I \in \mathfrak{F} \), we have that \( a = x \), \( x \in \mathfrak{F} \), if \( x \rightarrow 0 \).

Consequently, \( \sigma (L) \subseteq \mathfrak{F} \).

This establishes our earlier result: \( \sigma = \{ 2, 4 \} \) (only),\( \mathfrak{F} \), and \( a \) is isolated in \( \sigma (L) \)

We have that \( P = P^2 \) is an orthogonal projection. Also, if \( \sigma = \sigma _1 \cup \sigma _2 \),

\[
\sigma _1 = \{ 2 \} \quad \text{and} \quad \sigma _2 = \{ 4 \}.
\]

Hence \( P = P_{\sigma _1} + P_{\sigma _2} \). In addition,

\[
P_{\sigma _1} P_{\sigma _2} = 0.
\]

To see this,

\[
P_{\sigma _1}^2 = P_{\sigma _1} + P_{\sigma _2} + P_{\sigma _1} P_{\sigma _2} = P_{\sigma _1} + P_{\sigma _2} = P_{\sigma _1}.
\]
Because \( \sigma_1 = -\frac{1}{\tau} \int \frac{1}{\tau} (I-xI)^{-1} \, d\lambda \) and \( \frac{1}{\tau} \frac{d\sigma}{\tau} = \frac{1}{\tau} \frac{d\sigma}{\tau} \frac{d\lambda}{\tau} \), we have \( \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \). We also have

\[
P_0^2 = P_0 \Rightarrow \left( \sigma_1 + \sigma_2 \right)^2 = \sigma_1^2 + 2 \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_1 + \sigma_2 + \sigma_1 \sigma_2 + \sigma_2 \sigma_1
\]

\[
\Rightarrow \quad P_0 + P_0 = P_0 + P_0 + \sigma_1 \sigma_2 + \sigma_2 \sigma_1
\]

\[
\Rightarrow \quad 2P_0 \sigma_1 \sigma_2 = 2P_0 \sigma_1 = 0
\]

\[
\Rightarrow \quad P_0 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 P_0 = 0
\]

\[
\therefore \quad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_1 = 0
\]
Example: \( u_t = -u'' \), \( D(t) = \int_0^L u(x) \, dx \): 
\[ u'' + ku(1) = ku(0) \]
\[ u(0) = u(1) \]

Resolvent kernel — (Green's first)

\[
\begin{align*}
\phi(x-y) & \quad -G'' - 2G = 8\sin x & \quad G(0,y) = 0, \ G(1,y) = 0 \\
& \quad -G'' - 2G(x-y) = 8\sin x
\end{align*}
\]

Standard standard solution: Take homogeneity solution in \( x < y \), \( \psi(x) = \sin(\sqrt{x}) \)
\( \psi(x) \) at \( x > y \), \( \psi(x) = \sin(\sqrt{x-1}) \).

By (4.8)

\[
G(x,y,\lambda) = \begin{cases} 
\frac{\min(\sqrt{x+y}) \min(\sqrt{x})}{\lambda} & x < y \\
\frac{\min(\sqrt{x+y}) \max(\sqrt{x-1})}{\lambda} & x > y,
\end{cases}
\]

\[
W = \begin{vmatrix}
u_1 & u_2 \\ u_1' & u_2'
\end{vmatrix} = \begin{vmatrix}
\min(\sqrt{x+y}) & \min(\sqrt{x-1}) \\
\max(\sqrt{x+y}) & \max(\sqrt{x-1})
\end{vmatrix}
\]

\[
= -\sqrt{\lambda} \min(\sqrt{x})
\]

\[
G(x,y,\lambda) = \frac{1}{\sqrt{\lambda} \min(\sqrt{x})} \begin{vmatrix}
\min(\sqrt{x+y}) & \min(\sqrt{x-1}) & x < y \\
\max(\sqrt{x+y}) & \max(\sqrt{x-1}) & x > y
\end{vmatrix}
\]

Resolvent op. \( (L-\lambda I)^{-1} f = \int_0^1 G(x,y,\lambda) f(y) \, dy \).