Math 642
Apr. 8, 2020 (Expanded.)

Last time: We showed that $L = L^* \Rightarrow \sigma(L) \subseteq \mathbb{R}$ and started an example

Today: Spectral measures — at ~~least~~ least a start.

1. The example.

$$Lu = -u'', \quad D(L) = \{u \in L^2[0,1] : u'' \in L^2[0,1], \text{ and } u(0) = u(1) = 0\}.$$

— This means that we are using Dirichlet data. Also, $L = L^*$

Eigenvalues of L $\quad \lambda_n = n^2\pi^2, \quad n = 1, 2, 3, \cdots$.

Green's Funct. $\quad$ ~~Let~~ $\lambda \in \mathbb{C}$, $\lambda \neq \{n^2\pi^2\}_{n=1}^{\infty}$.

As we want to find a Green's function for ~~so such that~~ the resolvent —

$$R_\lambda(L) = (L - \lambda I)^{-1}.$$

This means that

$$R_\lambda(L) f = \int_0^1 G(x, y, \lambda) f(y)\, dy.$$

The ~~equa~~ As usual, the Green's function will satisfy

$$-G''(x,y) - \lambda G = \delta(x-y),$$

$$G(0,y) = 0, \quad G(1,y) = 0.$$

There is a formula for this in the text, namely (4.8) on page 147. Since I'm not ~~using time~~ using class time, I'll derive it.

Finding the Green's function

(a.) Homogeneous soln's to $-G'' - \lambda G = 0$, $x \neq y$,

$G(0,y) = 0$, $G(1,y) = 0$.

There are two linearly independent ~~solu~~ solutions,

$$u_1 = \sin(\sqrt{\lambda}\, x), \quad u_2 = \sin(\sqrt{\lambda}(1-x)).$$

Here, $u_1(0) = 0$, $u_2(1) = 0$. Thus,

$$G(x,y) = \begin{cases} A \cancel{\sin(\sqrt{\lambda} x)} \sin(\sqrt{\lambda}\, x), & x < y \\ B \sin(\sqrt{\lambda}(1-x)), & x > y. \end{cases}$$

(b). Continuity at $x = y$.

$$A \sin(\sqrt{\lambda}\, y) = B \sin(\sqrt{\lambda}(1-y)).$$

(c) Jump at $x = y$.

$$-G'(x,y)\Big|_{x=y^-}^{x=y^+} = 1$$

$$\Rightarrow \quad G'(y^-, y) - G'(y^+, y) = 1.$$

$$A\left(\sqrt{\lambda} \cos(\sqrt{\lambda}\, y)\right) - B\sqrt{\lambda}(-1)\cos(\sqrt{\lambda}(1-y)) = 1.$$

(a) Equations

$$A \sin(\sqrt{\lambda}\, y) - B \sin[\sqrt{\lambda}(1-y)] = 0$$

$$A \cos(\sqrt{\lambda}\, y) + B \cos(\sqrt{\lambda}(1-y)) = \frac{1}{\sqrt{\lambda}}$$

~~Solving~~ Solving these we get —

$$A = \frac{\sin(\sqrt{\lambda}(1-y))}{\sqrt{\lambda}\sin(\sqrt{\lambda})}$$

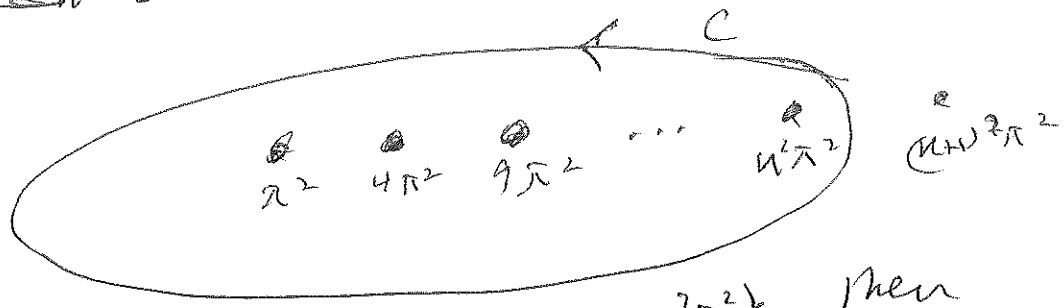
$$B = \frac{\sin(\sqrt{\lambda}\, y)}{\sqrt{\lambda}\sin(\sqrt{\lambda})}.$$

Solution

$$\therefore\ G(x,y,\lambda) = \frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda})}\begin{cases} \sin(\sqrt{\lambda}(1-y))\sin(\sqrt{\lambda}x), & x<y \\ \sin(\sqrt{\lambda}y)\sin(\sqrt{\lambda}(1-x)), & x>y. \end{cases}$$

Note: $G(x,y,\lambda)$ is ~~symmetric~~ symmetric in x & y, so we have $G(x,y,\lambda) = G(y,x,\lambda)$.

3. <u>Projections</u>. Let C be a simple closed curve encircling the eigenvalues up ~~to~~ to and including $n^2\pi^2$.



~~Let~~ Let $\sigma = \{\pi^2, 4\pi^2, \dots, n^2\pi^2\}$. Then we have

$$P_\sigma = -\frac{1}{2\pi i}\oint_C (L-\lambda I)^{-1}\,d\lambda,$$

In kernel form this is

$$P_\sigma(x,y) = -\frac{1}{2\pi i}\oint_C G(x,y,\lambda)\,d\lambda.$$

~~where x,~~

Fix x, y. Because $G(x,y,\lambda) = G(y,x,\lambda)$, $P_\sigma(x,y) = P_\sigma(y,x)$. So, lets work with $x<y$.

Thus,

$$P_\sigma(x,y) = -\frac{1}{2\pi i}\oint_C \frac{\sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}(1-y))}{\sqrt{\lambda}\,\sin(\sqrt{\lambda})}\,d\lambda.$$

We will use the residue theorem to evaluate this integral.

First, we need to identify the singularities ~~sin~~ in $\mathbb{C}$. These ~~occu~~ occur at the places where $\sqrt{\lambda}\sin(\sqrt{\lambda}) = 0$:

$$\lambda = 0,\ \lambda = \pi^2,\ \lambda = 4\pi^2, \ldots, \lambda = n^2\pi^2.$$

I claim that $\lambda = 0$ is a removable singularity.

(i) $$\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} = \frac{\sqrt{\lambda}x}{\sqrt{\lambda}}\left(1 - \frac{\lambda x^2}{3!} + \frac{\lambda^2 x^4}{5!} \cdots\right)$$

$$= x\left(1 - \frac{\lambda x^2}{3!} \cdots\right) \leftarrow$$ Analytic at $\lambda = 0$ and nonzero there.

(ii) $$\frac{\sin(\sqrt{\lambda}(1-y))}{\sin(\sqrt{\lambda})} = \frac{\sqrt{\lambda}(1-y)\left(1 - \frac{\lambda(1-y)^2}{3!} + \frac{\lambda^2(1-y)^4}{5!} \cdots\right)}{\sqrt{\lambda}\left(1 - \frac{\lambda}{3!} + \frac{\lambda^2}{5!} \cdots\right)}$$

$$= (1-y)\underbrace{\left\{\frac{\left(1 - \frac{\lambda(1-y)^2}{3!} + \frac{\lambda^2(1-y)^4}{5!} \cdots\right)}{\left(1 - \frac{\lambda}{3!} + \frac{\lambda^2}{5!} \cdots\right)}\right\}}_{\text{Analytic at } \lambda = 0 \text{ \& nonzero there}}$$

(iii) <u>Result:</u> $\dfrac{\sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}(1-y))}{\sqrt{\lambda}\sin(\sqrt{\lambda})}$ is analytic at $\lambda = 0$; the singularity is removable.

Second, we have to deal with the singularities at $k^2\pi^2$, $k=1,\dots,n$. At first glance, it looks ~~like~~ like $\sqrt{\lambda}$, which has a branch cut, might cause problems. In fact it doesn't. To see this, look at

$$\cancel{\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}} \quad \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} = \left(\sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{\lambda}x)^{2k+1}}{(2k+1)!}\right)\frac{1}{\sqrt{\lambda}}$$

$$= \frac{\sqrt{\lambda}x}{\sqrt{\lambda}}\left(\sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k x^{2k}}{(2k+1)!}\right)$$

$$= x\left(\sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k x^{2k}}{(2k+1)!}\right)$$

Thus $\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$ is really a function of λ, ~~Thus~~ rather than $\sqrt{\lambda}$. The branch cut ~~is thus nothing to do~~ isn't really there. The same thing happens to the other ~~term, the~~ term. There are simple poles. ~~The reason~~ The reason is that $\sin(\sqrt{\lambda})$ has a simple 0 at $\lambda=0$. Why?

$$\frac{d}{d\lambda}\left(\sin(\sqrt{\lambda})\right)\Big|_{\lambda=k^2\pi^2} = \frac{1}{2}\lambda^{-1/2}\cos(\sqrt{\lambda})\Big|_{\lambda=k^2\pi^2} = \frac{1}{2k\pi}\frac{\cos(k\pi)}{(-1)^k}.$$

Residues at $\lambda = k^2\pi^2$. We will ~~use~~ use this formula to evaluate the residues:

Let $h(z) = \frac{f(z)}{g(z)}$. If $g(z_0) = 0$ and $g'(z_0) \neq 0$, then z_0 is a simple zero. If in addition, $f(z_0) \neq 0$. Then $h(z)$ has a simple pole at $z = z_0$.

$$\cancel{\operatorname{Res}_{z_0}(h) = \lim_{z \to z_0} \left(\frac{f(z)}{g(z)} \right)} \quad \lim$$

$$\operatorname{Res}_{z_0}(h) = \lim_{z \to z_0} \left(\frac{(z - z_0) f(z)}{g(z)} \right) = \lim_{z \to z_0} \left(\frac{f(z)}{\frac{g(z) - g(z_0)}{z - z_0}} \right)$$

$$= \frac{f(z_0)}{g'(z_0)}.$$

Using this,

~~Using this~~

$$\operatorname{Res}_{\lambda = k^2\pi^2} \left(\frac{\sin(\sqrt{\lambda}x) \sin(\sqrt{\lambda}(1-y))}{\sqrt{\lambda} \sin(\sqrt{\lambda})} \right)$$

$$= \frac{\sin(\pi k x) \sin(\pi k(1-y))}{\pi k \left(\frac{1}{2k\pi} \cos(k\pi) \right)}$$

$$= \frac{2 \sin(\pi k x) (-1)^{k+1} \sin(\pi k y)}{(-1)^k}$$

$$= -2 \sin(\pi k x) \sin(\pi k y)$$

Hence,

$$\cancel{P_\sigma(x,y) = \frac{1}{2\pi i}\left(\sum_{k=1}^{n} \cdots\right)}$$

$$P_\sigma(x,y) = -\frac{1}{2\pi i} \times \left(\sum_{k=1}^{n} 2\pi i \, \mathrm{Res}_{\pi k}\left(\cdots\right)\right)$$

$$= -\left(-2 \sum_{k=1}^{n} \sin(\pi k x) \sin(\pi k y)\right)$$

$$= \sum_{k=1}^{n} 2 \sin(\pi k x) \sin(\pi k y).$$

~~Note: $P_\sigma f(x) =$~~

Note, if $f \in L^2[0,1]$, then

$$P_\sigma f(x) = \sum_{k=1}^{n} b_k \sin(\pi k x), \quad b_k = 2\int_0^1 \sin(\pi k y) f(y)\,dy.$$

What is this? Look at the ~~orth~~ orthogonal set

$$\cancel{\{\sin(x), \sin(2x), \ldots, \sin(nx)\}}.$$

$$\{\sin(\pi k x)\}_{k=1}^{\infty}.$$

P_σ is just the projection onto $\mathrm{sp}\{\sin(\pi k x), k=1,\ldots,n\}$

This is just the partial sum

$$P_\sigma f = f_n(x) = \sum_{k=1}^{n} b_k \sin(k\pi x).$$