Math 642
Apr. 8, 20 20 (Expanded)

Last time: We showed that $L=L^\infty \Rightarrow \sigma(L) \subseteq \mathbb{R}$ and
started an example.

Today: Spectral measures — at least a start.

1. The example,

$L u = -u''$, $\mathcal{D}(L) = \{ u \in L^2(\mathbb{R}) : u \in W^2_2(\mathbb{R}),$ and $u(0) = u(1) = 0 \}$.

This means that we are using Dirichlet data. Also, $L = L^\infty$.

As:

Eigenvalues: $L_n = n^2 \pi^2$, $n = 1, 2, 3, \ldots$.

Green's Function: $\lambda \in \mathbb{C}$, $\lambda \neq \frac{k^2 \pi^2}{3}$ $n = 1$.

As we want to find a Green's function for

such that

\[ R(\lambda) = (L - \lambda I)^{-1}. \]

This means that

\[ \int_R \frac{G(x, y)}{\lambda} dy = \int_0^1 \mathcal{G}(x, y) \delta(y) dy. \]

As usual, the Green's function will satisfy

\[ -G''(x, y) - \lambda G = \delta(x - y), \]

$G(0, y) = 0$, $G(1, y) = 0$. 


There is a formula for this in the text, namely (4.8) in page 177. Since I'm not using class time, I'll derive it.

Finding the Green's function

(a) Homogeneous solution to \(-G'' + G = 0, \quad x \neq y\),

\[ G(x, y) = 0, \quad G(1, y) = 0. \]

There are two linearly independent solutions,

\[ u_1 = \sin (\sqrt{\lambda} x), \quad u_2 = \sin (\sqrt{\lambda} (1-\lambda x)), \]

where, \( u_1(0) = 0, \quad u_2(0) = 0. \)

Thus,

\[ G(x, y) = \begin{cases} A \sin (\sqrt{\lambda} x) \sin (\sqrt{\lambda} (1-\lambda x)), & \forall \lambda \leq y \\ B \sin (\sqrt{\lambda} (1-\lambda x)), & \forall \lambda > y. \end{cases} \]

(b) Continuity at \( x = y \).

\[ A \sin (\sqrt{\lambda} y) = B \sin (\sqrt{\lambda} (1-\lambda y)). \]

(c) Jump at \( x = y \).

\[ -G^+(x, y) \bigg|_{x = y^-} = 1 \]

\[ \Rightarrow \quad G'(y^-, y) - G'(y^+, y) = 1. \]

\[ A \sqrt{\lambda} \cos (\sqrt{\lambda} y) - B \sqrt{\lambda} \cos (\sqrt{\lambda} (1-\lambda y)) = 1. \]
\[ (a) \text{ Equation} \]
\[ A \sin \left( \frac{\pi}{A} y \right) - B \sin \left( \frac{\pi}{B} (a-1)y \right) = 0 \]
\[ A \cos \left( \frac{\pi}{A} y \right) + B \cos \left( \frac{\pi}{B} (a-1)y \right) = \frac{1}{A} \]

Solving, then we set -
\[ A = \frac{\sin \left( \frac{\pi}{A} (1-y) \right)}{\overline{A} \sin \left( \frac{\pi}{A} \right)} \]
\[ B = \frac{\sin \left( \frac{\pi}{B} y \right)}{\overline{B} \sin \left( \frac{\pi}{B} \right)} \]

Solution
\[ C(x,y,\lambda) = \frac{1}{\overline{A} \overline{B} \overline{\lambda}} \left\{ \begin{array}{ll}
\sin \left( \frac{\pi}{A} (1-y) \right) \sin \left( \frac{\pi}{B} x \right), & 0 < y < 1 \\
\sin \left( \frac{\pi}{B} y \right) \sin \left( \frac{\pi}{A} (1-x) \right), & 0 < x < 1 \\
\end{array} \right. \]

Note: \( C(x,y,\lambda) \) is symmetric in \( x, y \), so we have \( C(x,y,\lambda) = C(y,x,\lambda) \).
3. **Projections.** Let $C$ be a simple closed curve enclosing the eigenvalues up to and including $\pi^2$.

Let $D = \{\pi^2, 4\pi^2, \ldots, n^2\pi^2\}$. Then we have

$$P_0 = -\frac{1}{2\pi i} \oint_C (L-\lambda) \, d\lambda.$$

In kernel form this is

$$P_0(x,y) = -\frac{1}{2\pi} \oint_C G(x,y,\lambda) \, d\lambda.$$

where $\lambda$.

Fix $x, y$. Because $G(x,y,\lambda) = G(y,x,\lambda)$,

$$P_0(x,y) = P_0(y,x).$$

So, let's start with $x < y$.

Thus,

$$P_0(x,y) = -\frac{1}{2\pi i} \oint_C \frac{\sin(\lambda x) \sin(\lambda y)}{\sqrt{\lambda} \sin(\lambda \pi)} \, d\lambda.$$
First, we need to identify the singularities in $C$. These occur at the places where $\Gamma_k \sin(\Delta) = 0$.

$$\Delta = \pi, \; \lambda = \pi^2, \; \Delta = \frac{\pi^2}{2}, \; \Delta = \frac{\pi^2}{3}, \; \ldots$$

I claim that $\lambda = 0$ is a removable singularity.

(i) \[ \frac{\sin(\Delta x)}{\Gamma_k} = \frac{\sin(\Delta x)}{\Delta x} \left( 1 - \frac{\lambda x^2}{3!} + \frac{\lambda^2 x^4}{5!} \ldots \right) \]

\[ = x \left( 1 - \frac{\lambda x^2}{3!} + \ldots \right) \quad \text{Analytic at } x = 0 \]

and nonzero there.

(ii) \[ \frac{\sin(\Delta (1-y))}{\sin(\Delta)} = \frac{\sin(\Delta (1-y))}{\Delta (1-y)} \left( 1 - \frac{\lambda (1-y)^2}{3!} + \frac{\lambda^2 (1-y)^4}{5!} \ldots \right) \]

\[ = (1-y) \left\{ \left( 1 - \frac{\lambda (1-y)^2}{3!} + \frac{\lambda^2 (1-y)^4}{5!} \ldots \right) \right\} \left( 1 - \frac{\lambda}{3!} + \frac{\lambda^2}{5!} \ldots \right) \]

Analytic at $\lambda = 0$.

Nonzero there.

(iii) Result: \[ \frac{\sin(\Delta x) \sin(\Delta (1-y))}{\Gamma_k \sin(\Gamma_k)} \text{ is analytic at } x = 0 \; \text{for all } y \in \mathbb{C}. \]

The singularity is removable.
Second, we have to deal with the

singularity at \( \pm \pi, \kappa = 1, -1, \ldots \). At

first glance, it looks like \( \Gamma \kappa \),

which has a branch cut, might cause

problems. In fact it doesn't. To see

this, look at

\[
\sin \left( \frac{\kappa \pi x}{\lambda} \right) = \frac{\sin \left( \frac{\kappa \pi x}{\lambda} \right)}{\frac{\kappa \pi x}{\lambda}} \left( \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{\kappa \pi x}{\lambda})^{2k+1}}{(2k+1)!} \right) \frac{1}{\frac{\kappa \pi x}{\lambda}}
\]

\[
= \frac{\Gamma \kappa}{\lambda} \left( \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k} x^{2k}}{(2k+1)!} \right)
\]

Thus, \( \frac{\sin \left( \frac{\kappa \pi x}{\lambda} \right)}{\frac{\kappa \pi x}{\lambda}} \) is really a function of \( \kappa \),

rather than \( \frac{\kappa \pi x}{\lambda} \). The branch cut

\( \Gamma \kappa \) isn't really

needed. The same thing happens to the

other terms. There are

simple poles. The reason is that

\( \sin \left( \frac{\kappa \pi x}{\lambda} \right) \) as a simple \( \kappa \) at \( x = 0 \), why?

\[
\frac{\partial}{\partial \kappa} \left( \sin \left( \frac{\kappa \pi x}{\lambda} \right) \right) = \frac{1}{2} \lambda^{-1/2} \cos \left( \frac{\kappa \pi x}{\lambda} \right) \left( \frac{1}{2 \kappa} \cos \left( \frac{\kappa \pi x}{\lambda} \right) \right)
\]

\[
= \frac{1}{2 \kappa} \cos \left( \frac{\kappa \pi x}{\lambda} \right)
\]

\[
\kappa = \pm \pi^2 \]
Residues at $z = b^2$.

We will use this formula to evaluate the residues:

$$\text{Res}_{z = z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z)}{g'(z)}$$

If $g(z_0) = 0$ and $g'(z_0) \neq 0$, then $z_0$ is a simple zero. If in addition, $f(z_0) \neq 0$, then $f(z)$ has a simple pole at $z = z_0$.

Res_{z_0} \left( \frac{f(z)}{g(z)} \right) = \lim_{z \to z_0} \frac{f(z)}{g(z)}

= \frac{f(z_0)}{g''(z_0)}.

Using this,

Thus

$$\text{Res}_{z = b^2} \left( \frac{\sin (\pi x) \sin (\pi(1-y))}{17 \pi \left| \sin \left( \frac{\pi}{2} \right) \right|} \right)

= \frac{\sin \left( \pi b x \right) \sin \left( \pi(1-y) \right)}{\lambda b \left| \sin \left( \frac{\pi}{2} \right) \right|}.

= \frac{2 \sin \left( \pi b x \right) \left( \frac{(-1)^{k+1} b^{k+2}}{(b+1)^{k+2}} \sin \left( \pi(1-y) \right) \right)}{(-1)^k}

= -2 \sin \left( \pi b x \right) \sin \left( \pi(1-y) \right)$$
Hence,

\[ P_0(x, y) = -\frac{1}{2\pi i} \sum_{k=1}^{n} 2ik \Re e \left( \frac{1}{\pi k x + iy} \right) \]

\[ = -\left( -2 \sum_{k=1}^{n} \frac{1}{\pi k x + iy} \sin(\pi k y) \right) \]

\[ = \sum_{k=1}^{n} 2 \sin(\pi k x) \sin(\pi k y). \]

Note: \[ \sum \]

Note, if \( f \in L^2(0, 1) \), then

\[ P_0 f(x) = \sum_{k=1}^{n} b_k \sin(\pi k x), \quad b_k = 2 \int_0^1 f(y) \sin(\pi k y) dy. \]

What is \( P_0 \)? Look at the inner product:

\[ \langle \sin(\pi k x), \sin(\pi k x) \rangle, \quad k = 1, 2, \ldots \]

\( P_0 \) is just the projection onto \( \mathbb{R} \{ \sin(\pi k x), k = 1, \ldots, n \} \).

\( P_0 f = f \), is just the partial sum

\[ P_0 f = f_n(x) = \sum_{k=1}^{n} b_k \sin(\pi k x). \]