# Final Examination

This take-home exam is due at 12 noon on Wednesday, May 6, 2020. You may consult any written or online source. You may *not* consult any person, either a fellow student or faculty member, except your instructor

1. **(15 pts.)** Let $L = L^*$ be a self-adjoint operator defined on a domain $D$. Show that if $\langle Lf, f\rangle \ge 0$ for all $f \in D$, then all $\lambda$ such that $\lambda \notin [0,\infty)$ are in the resolvent set $\rho(L)$.

2. Consider the self adjoint operator $Lu = -u''$, $0 < x < \infty$, with $D_L = \{u \in L^2([0,\infty))\colon u'' \in L^2([0,\infty)),\ u'(0) = 0\}$.

   (a) **(10 pts.)** Find the Green's function for $L$.

   (b) **(15 pts.)** Use Stone's formula to find the associated spectral transform.

3. Let $L = L^*$ be a self-adjoint operator defined on a domain $D$ and let $E_\lambda$ be the right-continuous spectral family associated with $L$.

   (a) **(5 pts.)** Show that if $\mu \le \lambda$, then $E_\lambda E_\mu = E_\mu E_\lambda = E_\mu$ is equivalent to $E_\mu \le E_\lambda$.

   (b) **(5 pts.)** Show that $P = E_\lambda - E_\mu$ is and orthogonal projection.

   (c) **(10 pts.)** Suppose that $E_\lambda - E_{\lambda^-} \ne 0$. Show that $\lambda$ is an eigenvalue of $L$.

4. Suppose $f$ is a band-limited signal, with $\widehat{f}(\omega) = 0$ for $|\omega| \ge \Omega$. For $a > 1$, let $g_a = \mathcal{F}^{-1}(\widehat{g}_a)$, where $\widehat{g}_a$ is the function whose graph is given by Figure 1. Suppose that let $f$ is sampled at the rate $a\Omega/\pi$ which is higher that the Nyquist rate, $\Omega/\pi$. This means that we are oversampling $f$.

   (a) **(5 pts.)** Show that, on $|\omega| \le a\Omega$, $\widehat{f}(\omega)$, which is 0 for $|\omega| \ge \Omega$, can be represented by the $2a\Omega$ periodic Fourier series,

$$\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-in\pi\omega/a\Omega} \quad \text{with } c_{-n} = \frac{\pi}{a\Omega} f\Big(\frac{n\pi}{a\Omega}\Big), \qquad (1)$$

   In addition, $\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-in\pi\omega/a\Omega}\, \widehat{g}_a(\omega)$, on $[-a\Omega, a\Omega]$, because $\widehat{g}_a(\omega) = 1$ on $[-\Omega, \Omega]$ and $\widehat{f}(\omega) = 0$ on $|\omega| \ge \Omega$.

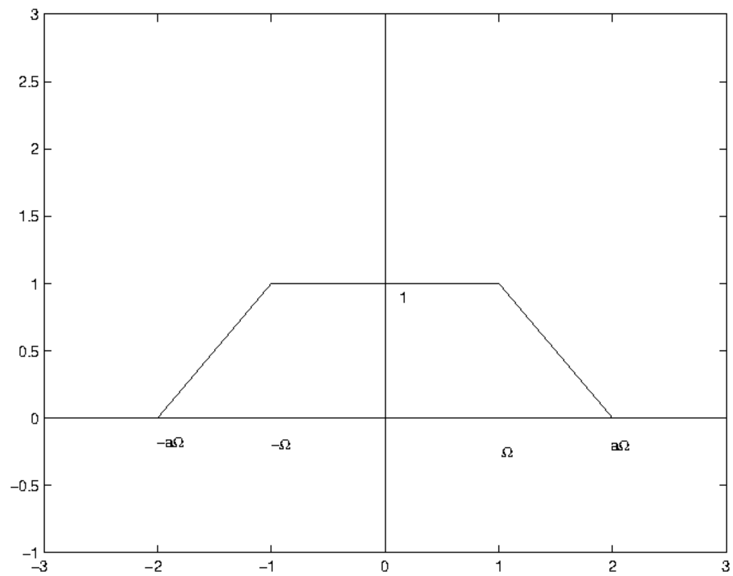


Figure 1: Graph of $\widehat{g}_a$

(b) **(5 pts.)** Show that

$$g_a(t) = \frac{\cos(\Omega t) - \cos(a\Omega t)}{\pi(a-1)\Omega t^2} \tag{2}$$

(c) **(5 pts.)** Use the previous two parts to show that

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\pi}{a\Omega} f(\frac{n\pi}{a\Omega}) g_a(t - \frac{n\pi}{a\Omega}). \tag{3}$$

(d) **(10 pts.)** Applying Parseval's theorem to the Fourier series (1), show that $\sum_{n\in\mathbb{Z}} |f(\frac{n\pi}{a\Omega})|^2 = \frac{a\Omega}{\pi}\|f\|^2_{L^2(\mathbb{R})}$. For all $|t| \le \frac{N\pi}{2a\Omega}$, show that $|f(t) - \sum_{|n|\le N-1} \frac{\pi}{a\Omega} f(\frac{n\pi}{a\Omega}) g_a(t - \frac{n\pi}{a\Omega})| \le C\|f\|_{L^2(\mathbb{R})} N^{-3/2}$. What is the rate for the standard sampling series? Does oversampling improve the rate of convergence?

5. **(15 pts.)** Prove this version of Watson's lemma: Suppose that $F(x) := \int_{-\infty}^{\infty} f(t) e^{-xt^2} dt$, where $f(t) = \sum_{n=0}^{\infty} a_n t^n$ uniformly in $|t| \le T$. In addition, for $|t| \ge T$, there is an $\alpha > 0$ such that $|f(t)| \le |t|^\alpha$. Show that

$$F(x) \sim \sum_{k=0}^{\infty} a_{2k} \Gamma(k + \frac{1}{2}) x^{-k-\frac{1}{2}}, \ x \to \infty.$$