

APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

January 11, 2021

Applied Analysis Part, 2 hours

Name: _____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: Do any three problems. Show all work clearly. State the problem that you are skipping. No extra credit for doing all four.

Problem 1. Recall that the DFT and inverse DFT are given by $\hat{y}_k = \sum_{j=0}^{n-1} y_j \bar{w}^{jk}$ and $y_j = \frac{1}{n} \sum_{k=0}^{n-1} \hat{y}_k w^{jk}$, where $w = e^{2\pi i/n}$.

- (a) State and prove the Convolution Theorem for the DFT.
- (b) Let a, x, y be column vectors with entries $a_0, \dots, a_{n-1}, x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}$. In addition, let α, ξ and η be n -periodic sequences, the entries for one period, $k = 0, \dots, n - 1$, being those of a, x , and y , respectively. Consider the circulant matrix

$$A = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}.$$

Show that the matrix equation $Ax = y$ is equivalent to convolution $\eta = \alpha * \xi$.

- (c) What are the eigenvalues and eigenvectors of A ? Use parts (a) and (b) to prove your answer.

Problem 2. Let $Lu = -\frac{d^2u}{dx^2}$, $0 \leq x \leq 1$, with the domain of L given by

$$D_L := \{u \in L^2[0, 1] : u'' \in L^2[0, 1], u(0) = -u(1), u'(0) = -u'(1)\}.$$

- (a) Show that L is self adjoint on $D(L)$.
- (b) Find the Green's function $G(x, y)$ for the problem $Lu = f$, $u \in D_L$.
- (c) Show that $Ku := \int_0^1 G(\cdot, y)u(y)dy$ is a compact self-adjoint operator.
- (d) Without actually finding them, show that the eigenfunctions of L contain an orthonormal set that is complete in $L^2[0, 1]$.

Problem 3. Let \mathcal{H} be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on \mathcal{H} .

- (a) State and prove the Closed Range Theorem.
- (b) Let $\mathcal{H} = L^2[0, 1]$. Define the kernel $k(x, y) := x^3y^2$ and let $Ku(x) = \int_0^1 k(x, y)u(y)dy$. Show the K is in $\mathcal{C}(L^2[0, 1])$.

- (c) Let $L = I - \lambda K$, $\lambda \in \mathbb{C}$, with K as defined in part (b) above. Find all λ for which $Lu = f$ can be solved for all $f \in L^2[0, 1]$. For these values of λ , find the resolvent $(I - \lambda K)^{-1}$.

Problem 4. Consider the functions ϕ and ψ defined below:

$$\phi(x) = \begin{cases} (|x| - 1)^2(2|x| + 1) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}, \quad \psi(x) = \begin{cases} x(|x| - 1)^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$

- (a) Let $n \geq 2$ and $0 \leq j \leq n$. Show that the functions $\phi_j(x) := \phi(nx - j)$ and $\psi_j(x) := \frac{1}{n}\psi(nx - j)$ satisfy the following: $\phi_j(k/n) = \delta_{j,k}$, $\phi'_j(k/n) = 0$, $\psi_j(k/n) = 0$ and $\psi'_j(k/n) = \delta_{j,k}$.
- (b) Use part (a) to show that the set $\{\phi_j, \psi_j\}_{j=0}^n$ forms a basis for the finite element space $S^{\frac{1}{3}}(\mathfrak{3}, 1)$. You may assume that $\dim S^{\frac{1}{3}}(\mathfrak{3}, 1) = 2n + 2$.
- (c) Use part (b) to define a (non-orthogonal) projection on $C^{(1)}[0, 1]$.