The Structure of the Retarded Field of an Eternally Uniformly Accelerated Charge, and Its Implications for the Existence of Radiation and the Principle of Equivalence

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1. The major steps in the understanding of the uniformly accelerated electric charge are the works of Born [1], Schott [2], Bondi & Gold [3], Rohrlich [4], and Boulware [5]. For simplicity I'll work mostly with the scalar analog of Boulware's work, presented by Ren & Weinberg [6].

2. Consider a scalar field satisfying the massless Klein–Gordon equation with a scalar source in place of the electromagnetic charge-current density. Let the source q (also loosely called "charge") move along the standard hyperbolic trajectory, $x_{\mathrm{s}}(s)$, with acceleration a,

$$t_{\mathrm{s}} = a^{-1}\sinh(as), \quad z_{\mathrm{s}} = a^{-1}\cosh(as), \quad x = 0 = y. \tag{1}$$

The charge should have a constant magnitude in its rest frame; Lorentz contraction of the volume then dictates that in the lab frame,

$$\rho(t,\mathbf{x}) = q\sqrt{1-\mathbf{v}(t)^2}\,\delta(\mathbf{x}-\mathbf{x}_{\mathrm{s}}(s(t))) \tag{2a}$$

$$= \frac{q}{a\sqrt{t^2+a^{-2}}}\,\delta(x)\delta(y)\delta(z-\sqrt{t^2+a^{-2}}). \tag{2b}$$

The wave equation is

$$\Box\phi = -\rho. \tag{3}$$

Let

$$r^2 = x^2+y^2. \tag{4}$$

Then the retarded solution can be constructed from the retarded Green function of a space-time point source as

$$\phi(x) = \frac{q}{2\pi}\int_{-\infty}^{\infty} ds\,\theta(t-x_{\mathrm{s}}^0(s))\delta([x-x_{\mathrm{s}}(s)]^2), \tag{5}$$

where x and $x_{\rm s}$ are the obvious 4-vector notations for the argument of ϕ and the position of the source, (1).

3. The integral (5) evaluates (see the next section) to

$$\phi(t,x,y,z) = \frac{q}{4\pi R}\,\theta(t+z), \tag{6}$$

where

$$R = \frac{a}{2}\sqrt{(z^2 - t^2 + r^2 - a^{-2})^2 + 4a^{-2}r^2}. \tag{7a}$$

It looks like a Coulomb field somehow distorted by the acceleration. Although a static Coulomb field extends as $|\mathbf{x} - \mathbf{x}_s|^{-1}$ all the way out to spatial infinity, ϕ on any time slice does not extend beyond the past horizon, $t + z = 0$. (Boulware, pp. 178–180, shows how this comes about in the limit of eternal acceleration.) Formulas equivalent to (7a) are

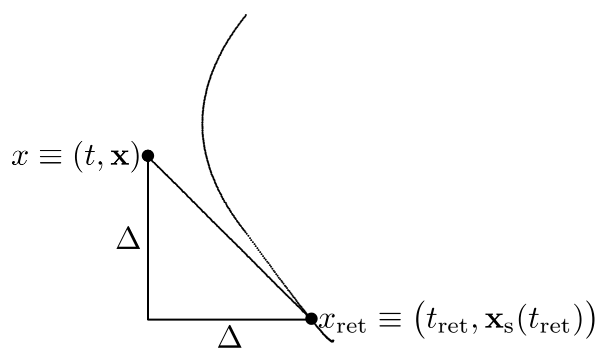
$$\begin{aligned} R &= \frac{a}{2}\sqrt{(z^2 - t^2 + r^2 + a^{-2})^2 - 4a^{-2}(z^2 - t^2)} && (7b) \\ &= \frac{a}{2}\sqrt{(z^2 - t^2 - r^2 - a^{-2})^2 + 4r^2(z^2 - t^2)}. && (7c) \end{aligned}$$

All three versions are valid wherever $t+z > 0$, even though the sign of $z^2 - t^2$ changes upon crossing the future horizon at $t - z = 0$ and the sign of $z^2 - t^2 - a^{-2}$ changes upon crossing the hyperbola.

4. To better understand R geometrically, let

$$\mathbf{r} = \mathbf{x} - \mathbf{x}_{\rm s}(t_{\rm ret}), \quad \Delta = |\mathbf{r}|, \quad t_{\rm ret} = t - \Delta, \quad \mathbf{v} = \mathbf{v}(t_{\rm ret}). \tag{8}$$

For a case when both space-time points are in the time-space plane of the paper, this notation is illustrated in this figure:



Then as shown in the appendix of [6], the Liénard–Wiechert potential of a point scalar source (with acceleration not necessarily uniform) is $q/4\pi R$ with

$$R = -(x^\mu - x^\mu_{\rm ret})U_\mu = \frac{\Delta - \mathbf{v}\cdot\mathbf{r}}{\sqrt{1 - \mathbf{v}^2}}. \tag{9}$$

It remains to show that (9) and (7) are equal for the trajectory (1). It's easy to see that (9) follows from (5), so this task reduces to the promised verification that (5) implies (6). Note first that on the hyperbola, $z_{\text{ret}} = \sqrt{t_{\text{ret}}^2 + a^{-2}}$ and

$$|\mathbf{v}| = \tanh(as) = \frac{t_{\text{ret}}}{z_{\text{ret}}}, \quad 1 - \mathbf{v}^2 = \frac{z_{\text{ret}}^2 - t_{\text{ret}}^2}{z_{\text{ret}}^2} = \frac{1}{a^{-2}(t_{\text{ret}}^2 + a^{-2})}, \tag{10}$$

which has already been used in (2). I shall now check a special case of (7). Suppose that $z = z_{\text{ret}}$, so that the null ray should be coming out of the plane of the figure. Then $\mathbf{v}\cdot\mathbf{r} = 0$, $t - t_{\text{ret}} = \Delta = r$, so by (9) $R = \gamma_{\mathbf{v}} r = ar\sqrt{t_{\text{ret}}^2 + a^{-2}}$. And that is what (7a) reduces to in this case after several steps of algebra. So far I have not succeeded in a similar calculation for the general case, or even the opposite special case, where $r = 0$ and (7) reduces to the simple form $R = \frac{a}{2}|z^2 - t^2 - a^{-2}|$. Howver, I'll derive (7) by another method in Sec. 6.

5. I have avoided Rindler coordinates so far, to emphasize that (7) does not depend on quadrant and can be obtained from a purely Minkowskian point of view. Let us now introduce Rindler coordinates in the region R (where $z > |t|$) by

$$t = \sigma \sinh(a\tau), \quad z = \sigma \cosh(a\tau). \tag{9}$$

Then $z^2 - t^2 = \sigma^2$, and R depends on t and z only through the combination σ; it is independent of the Rindler time. Similarly, in F (where $t > |z|$), let

$$t = \sigma \cosh(a\tau), \quad z = \sigma \sinh(a\tau). \tag{10}$$

Again R depends only on σ (and r), but now $\sigma^2 = t^2 - z^2$ and σ is a time coordinate.

6. I finally found out how to derive (7) directly from (5), following the approach in Boulware [5], pp. 176–177. As he says, it's easier to describe using the language of Rindler coordinates, which means one needs to treat quadrants R and F separately. I'll do only the former here. Since the theta function is unchanged when its argument is replaced by an increasing function of itself, we can write (5) as

$$\phi = \frac{q}{2\pi}\int ds\, \theta(\tau - s)\delta\big(r^2 + (z - z_{\text{ret}})^2 - (t - t_{\text{ret}})^2\big), \tag{10}$$

where the retarded point is parametrized as in (1). Thus

$$\begin{aligned}
\phi &= \frac{q}{2\pi}\int_{-\infty}^{\tau} ds\, \delta\big(r^2 + z^2 - t^2 + a^{-2}\cosh^2(as) - a^{-1}\sinh^2(as) \\
&\qquad\qquad - 2za^{-1}\cosh(as) + 2ta^{-1}\sinh(as)\big) \\
&= \frac{q}{2\pi}\int_{-\infty}^{\tau} ds\, \delta\big(r^2 + \sigma^2 + a^{-2} - 2\sigma a^{-1}\cosh(a\tau - as)\big).
\end{aligned}$$

To change the argument in the delta function to s we must divide by the old argument's derivative, $2\sigma\sinh(a\tau - as)$, and then evaluate at the point where the old argument vanishes,

$$\cosh(a\tau - as) = \frac{r^2 + \sigma^2 + a^{-2}}{2\sigma a^{-1}}\,.$$

That leads to

$$\sinh(a\tau - as) = \frac{\sqrt{(r^2 + \sigma^2 + a^{-2})^2 - 4\sigma^2 a^{-1}}}{2\sigma a^{-1}}$$

and hence

$$\phi = \frac{q}{2\pi}\,\frac{1}{a}\,[(r^2 + \sigma^2 + a^{-2})^2 - 4\sigma^2 a^{-2}]^{-1/2}\,,$$

which is $q/4\pi R$ with R in form (7b).

7. We can now catalog some properties of the function ϕ.

   (1) It indeed satisfies the wave equation (3) (with a source (2) on the hyperbola), even on the horizon where the theta function sits. Normally, "chopping" a solution by a theta function would introduce into the partial differential equation terms proportional to the delta function and its derivative, thereby violating the PDE. In this case we shall see that those terms vanish. However, nontrivial delta-function terms do appear in the first derivatives of ϕ. The true electromagnetic counterpart of ϕ is not the **E** and **B** fields, but the vector potential. Conversely, the scalar counterpart of **E** and **B** is the gradient of ϕ, whose components appear bilinearly in the components of the stress-energy-momentum tensor. So there is no conflict here with Bondi and Gold [3], and no real loss of smoothness in passing from scalar to EM.

   Proof of claim: It is useful to introduce the null coordinates

$$v = t + x, \quad u = t - x. \tag{11}$$

   In the metric signature with $g_{00} < 0$, the wave equation (3) becomes

$$4\,\frac{\partial^2\phi}{\partial v\,\partial u} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = \rho. \tag{12}$$

   Consider the homogeneous equation, $\Box\psi = 0$, and study $\phi = \psi\theta(v)$. For $\mu = t$ or x,

$$\partial_\mu{}^2\phi = \partial_\mu{}^2\psi\,\theta(v) + 2\partial_\mu\psi\,\delta(v) + \psi\,\delta'(v),$$

   so

$$\Box\phi = 2(\partial_z - \partial_t)\psi\,\delta(v) = 4\partial_u\psi\,\delta(v). \tag{13}$$

   So ϕ is a solution if and only if

$$\frac{\partial\psi}{\partial u} = 0 \quad \text{when } v = 0. \tag{14}$$

From (7) we see that $\psi = q/4\pi R$ depends on v and u only through the combination $z^2 - t^2 = -vu$. Therefore, its u derivative contains an overall factor of v, which forces (14) to hold.

(2) It is indeed retarded, because of the theta function. Because of the idealization of eternal acceleration, however, this solution just barely qualifies as retarded. One might think that a solution ϕ is retarded only if, at every point x where $\phi(x) \neq 0$, there is a backward-directed null ray that intersects the support of the source, ρ. The past horizon is in the support of ϕ, and its derivatives, which represent the intensity of the field, actually have a delta-function singularity there. But that ray misses the source; it does not intersect the support of ρ unless one passes to the conformal completion of the space-time. Nevertheless, ϕ is as retarded as any solution of (3) cum (2) can be, since any solution of the homogeneous wave equation that vanishes everywhere below the horizon must be a function of v alone and can't cancel the horizon singularity, nor satisfy the retarded condition everywhere above the horizon.

(3) In region R, ϕ is static, in the Rindler sense — i.e., independent of the time coordinate τ. It is also invariant under time reversal. It is hard to discern any radiation in this region, especially from the point of view of an observer moving along another hyperbolic orbit of the same Rindler Killing vector. This confirms Rohrlich's dictum that a coaccelerating observer sees no radiation.

(4) In region F, ϕ is dynamical — a function of time but not of the spatial coordinate (which is now τ, by my definition). So there may be "radiation" in F, but we need to be careful about what this means. For me the easy way out would be to cite the paper of Landulfo et al. [7], which breaks up ϕ into Unruh normal modes and shows that the amplitudes coincide with the results of an S-matrix analysis in the quantized theory. Traditionally the classical theory has been analyzed in terms of energy fluxes (the Poynting vector in the EM case). The most up-to-date treatments I know of are Sec. IV of [5] for the EM case and Sec. II of [6] for the scalar case. In a nutshell, Ren and Weinberg (following Bourlware) do two calculations. First, they integrate T^{tj} over a sphere on the future light cone of the charge and get the analog of Larmor's formula, $q^2a^2/(12\pi)$. [Sharp-eyed readers may object that this conclusion is inconsistent with my points 3 and 5 if the sphere is so small that it fits inside R. That is a tangled issue that I don't want to discuss right now.] Second, they do a calculation of the total energy flow through a certain space-time region entirely within R, and traversed by the charge's worldline, and get zero. This demonstrates that there is no contradiction between inertial and accelerated calculations inside R, but it leaves a murky issue of where the incoming energy comes from. Again, today is not the time to get into a critical analysis. I urge everyone to read the two papers, and, if possible, also Rohrlich's [4(b)].

(5) We could have constructed an advanced solution, which is nonzero in R and P only and has its theta function on the future horizon, $\theta(z - t)$. Then

$$\phi[\text{retarded}] = \phi[\text{advanced}] \quad \text{in the interior of R!} \tag{15}$$

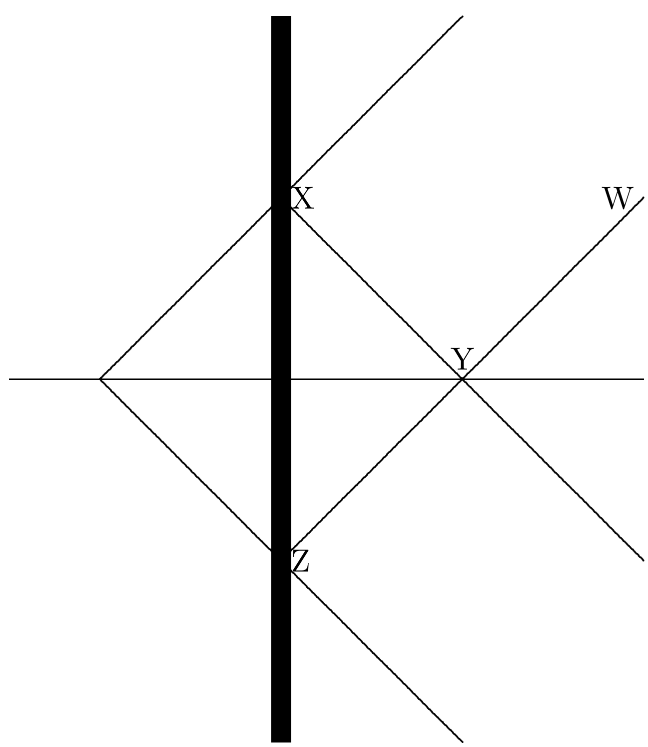
The time-symmetric function $(\phi[\text{ret}] + \phi[\text{adv}])/2$ is also a solution of the same PDE; it is zero only in L and has step singularities on both horizons. A different construction is to start with our unique ϕ in the interior of R, continue it smoothly across the future horizon into F as $\phi[\text{ret}]$, but also continue it smoothly across the past horizon into P as $\phi[\text{adv}]$. This is a smooth solution throughout the three regions; note that in F and P it differs from the symmetrized solution by a factor 2. Continuing this solution further, across the distant half-horizons into L, leads finally to disaster: it cannot satisfy the free wave equation in L; rather it has a source (with a negative sign) on the hyperbola that is a mirror image of the trajectory of the original source. In other words, as Boulware says, "The field in region [F] may be regarded as *either* the field due to a uniformly accelerated charge, [q], in region [R] *or* as the field due to a uniformly accelerated charge [$-q$] in region [L]; no measurements restricted to [F] can ever distinguish the two situations." [His similar comment about the fact $\phi[\text{ret}] = \phi[\text{adv}]$ in R is, "The field at any point in [R] may be interpreted *either* as the Coulomb field plus *outgoing* radiation field at the retarded time, *or* as the Coulomb field plus *incoming* radiation at the advanced time." This refers of course to the fact that there are a unique backward null line and a unique forward null line from the field point to the hyperbolic trajectory (see [5], Fig. 4).

8. Let me finish this overlong essay with an introduction to the controversy over the principle of equivalence. That issue will surely merit much deeper discussion in the future. The point is illustrated very well by p. 1507 of the 1999 paper of Pauri and Vallisneri [8], which I've placed at `https://www.math.tamu.edu/ fulling/ar/PVfig.pdf`. The radiated circular waves there indicate what I'll call the Rohrlich orthodoxy: No radiation is detected if the source and detector are both at rest (of course) or if they are coaccelerating (as per my point 7(3)). These are cases 4 and 1, respectively, in [8]'s terminology. If the detector is in free fall and the charge is accelerated (case 3), radiation is detected. (Cf. point 7(4), where the same conclusion is reached at least when the detector is far away from the source.) Finally, if the charge is in free fall and the detector accelerated (case 2), by symmetry one would expect radiation to be detected. However, Rohrlich and Pauri–Vallisneri notwithstanding, this conclusion is not universally accepted, to put it mildly. Many physicists consider it blatantly absurd, and respond with either "This proves that the principle of equivalence simply does not apply to electromagnetic phenomena!" or "This shows that the other conclusion must also be false: Surely a charge at rest on a table in my lab is not radiating, even if I'm falling while I'm looking at it."

9. In favor of the Rohrlich doctrine for the controversial case 2, one can cite parallel evidence from other scenarios. An accelerated Unruh detector is analogous to case 2, if one regards the detector as the "observer". On the other hand, if the detector is regarded as an emitter seen by an inertial observer [9], the analogue is case 3. For this system both radiation phenomena are now generally accepted, with, if anything, case 3 being slightly more controversial than case 2. Similarly, Scully et al. (2018)

[9] find that an atom freely falling into a black hole radiates as seen by a supported observer, provided that the quantized field is in a Boulware-like state; Svidzinsky et al. (2018) [10] find the analogous phenomenon for a static atom in flat space if a Rindler-like field state is created (by a uniformly accelerated mirror). Finally, Fulling and Wilson [11] (2019) carried the analogy down to a two-dimensional conformal scalar field interacting with a static mirror, which radiates if the field starts in a Rindler-like vacuum. In all these systems it seems that a qualitative version of the equivalence principle is satisfied and real radiation detection occurs. However, their relation to classical bremsstrahlung is muddied by the need to put the quantum field into some generalized "Rindler vacuum" when the emitter is in free fall. The distinction between Rindler (or Boulware) vacuum and Minkowski (or Unruh or Hartle–Hawking) vacuum has no counterpart in classical theory.

10. On the other hand, upon rereading I find Rohrlich's argument for case 3 unconvincing. We have seen that the crucial point about case 2 (and its consistency with the null result for case 4) is that experiment 4 takes place entirely within Rindler space, R, whereas the radiation in experiment 2 is observed in region F. Clearly, the relation between Rindler and Minkowski space is not reciprocal! In case 3 there is no analog of F. Furthermore, from the point of view of a Rindler observer the Coulomb field of a stationary charge is never completely behind a Rindler horizon, even if the charge itself is. Therefore, that field is not truly "retarded" in the way that the Ren-Weinberg field is. This seems to cause problems for Rohrlich's argument (see [4(c)], p. 182 and the long footnote extending onto p. 183). A detailed critique unfortunately is beyond the scope of this essay.



A slight elaboration: As remarked in point 3, not only the accelerated charge but also its Coulomb cloud is contained entirely in R. A static charge from the point of view of a Rindler observer has a trajectory whose initial and final infinite segments are outside the Rindler chart and whose middle part is asymptotic to “horizons” YX and YZ (see figure above). However, its Coulomb cloud slops over into the region immediately to the right of Z, whence it can causally influence events below the line ZYW.

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