What does a finite Poynting flux at infinity imply in case of a uniformly accelerated charge?

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There are many arguments why a uniformly accelerated charge may not be emitting radiation. The main three being:

In the case of a uniformly accelerated charge in the instantaneous rest frame, the magnetic field vanishes globally, meaning thereby a nil Poynting vector everywhere and therefore no radiation.

The radiation reaction is directly proportional to the rate of change of acceleration, implying that a uniformly accelerated charge `feels' no radiation reaction.

From the strong principle of equivalence, a uniformly accelerated charge is equivalent to a charge permanently stationary in a gravitational field, and such a completely time-static system could not be radiating power at all.

A finite Poynting flux at infinity

On the other hand, if one computes the Poynting flux at large distances from the charge in inertial frames other than the instantaneous rest frame of the charge, then one invariably finds a finite Poynting flux which is taken as evidence that there *is* radiation emitted by a uniformly accelerated charge.

The electromagnetic field of a uniformly accelerated charge

For an assumed one dimensional motion with a uniform proper acceleration, $\mathbf{a} = \gamma^3 \dot{\mathbf{v}}$, the " present" velocity $\mathbf{v}_o$ at time t is related to its value $\mathbf{v}_r$ at the retarded time $t_r = t - r/c$ as

$$\gamma_0 \mathbf{v}_0 = \gamma_r \mathbf{v}_r + (t - t_r)\mathbf{a} = \left[\gamma \mathbf{v} + \frac{r \gamma^3 \dot{\mathbf{v}}}{c} \right]_{t_r} , \qquad (1)$$

where γ_0, γ_r are the corresponding Lorentz factors.

Using Eq. (1) we get the magnetic field of a uniformly accelerated charge as

$$\mathbf{B} = \left[\frac{-e\mathbf{n} \times \gamma_0 \mathbf{v}_0 / c}{\gamma^3 r^2 (1 - \mathbf{n} \cdot \mathbf{v}/c)^3} \right]_{t_r} . \qquad (2)$$

Thus we find that the magnetic field vector in the case of a uniformly acceleration is at any space-time event is proportional to the 'present' velocity $\mathbf{v}_0$ of the charge, how-so-ever far it may be from the charge location.

In the instantaneous rest-frame, where, by definition, the present velocity $\mathbf{v}_0 = 0$, we have a nil magnetic field, $\mathbf{B} = 0$ *everywhere*.

While the electric field of a uniformly accelerated charge is

$$\mathbf{E} = \left[\frac{e\mathbf{n}}{\gamma^2 r^2(1-\mathbf{n}\cdot\mathbf{v}/c)^2} + \frac{e\mathbf{n}\times(\mathbf{n}\times\gamma_0\mathbf{v}_0/c)}{\gamma^3 r^2(1-\mathbf{n}\cdot\mathbf{v}/c)^3}\right]_{t_r} . \tag{3}$$

The transverse components are proportional to the 'present' velocity $\mathbf{v}_0$ of the charge, and the fields fall as $1/r^2$, irrespective of its distance from the charge location, and becomes zero, when $\mathbf{v}_0 = 0$ at $t = 0$.

As the uniformly accelerated charge at time t is located at $z_e = [z_0^2 + c^2t^2]^{1/2}$, we can rewrite the fields with respect to z_e as

$$\begin{aligned} E_z &= -4ez_0^2(z_e^2 + \rho^2 - z^2)/\xi^3 \\ E_\rho &= 8ez_0^2\rho z/\xi^3 \\ B_\phi &= 8ez_0^2\rho ct/\xi^3 \ , \end{aligned} \tag{4}$$

with

$$\xi = [(z_e^2 - \rho^2 - z^2)^2 + 4z_0^2\rho^2]^{1/2}.$$

These 'real-time' field expressions can be derived from the time-retarded field expressions by an algebraic transformation[1].

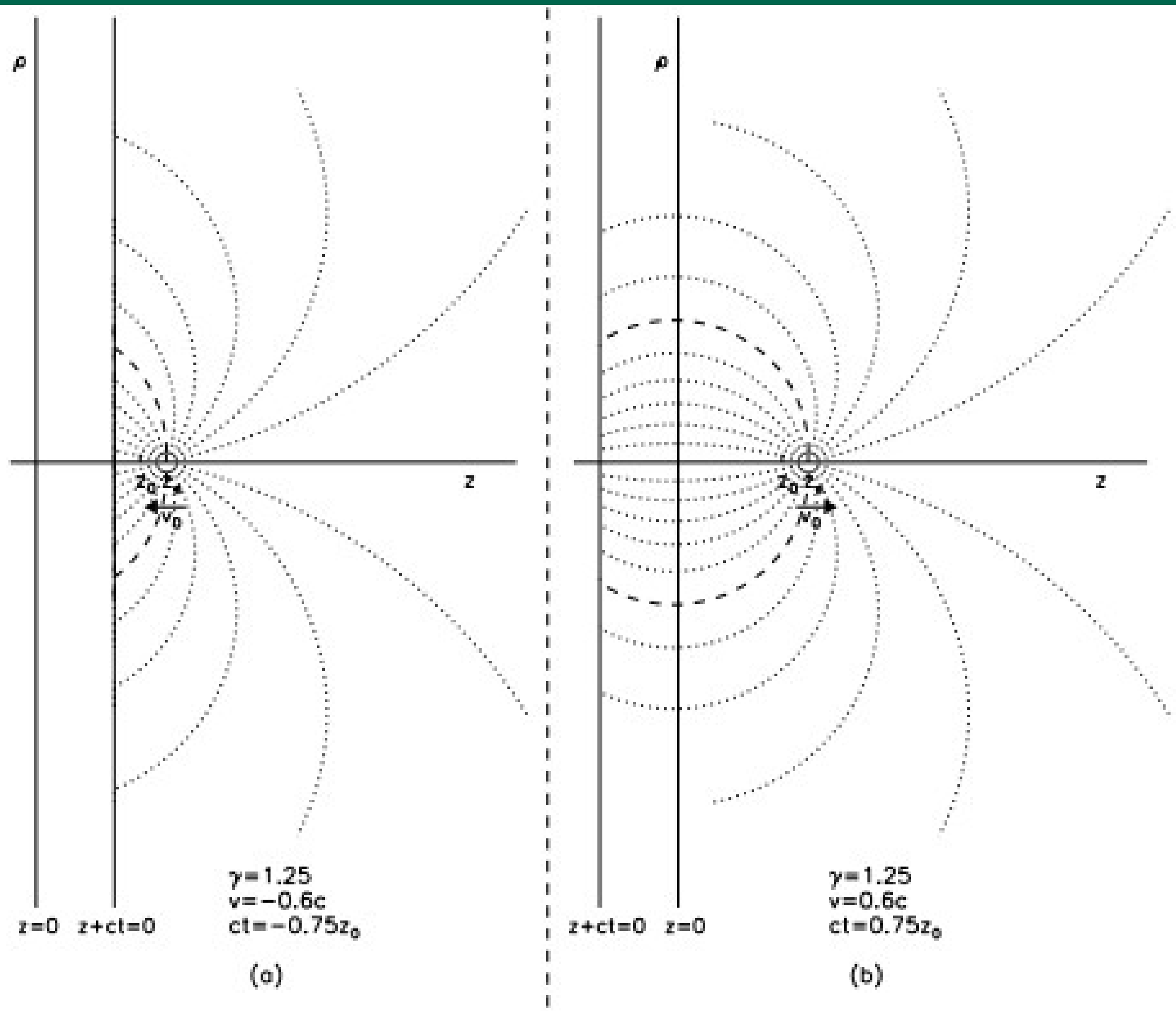

FIG. 1. Electric field lines of a charge, moving with a uniform proper acceleration along the z-axis. The field lines are drawn for a chosen time (a) $t = -0.75z_0/c$ when the charge was moving with a velocity $v = -0.6c$, (b) at the time $t = 0.75z_0/c$ when the charge was moving with a velocity $v = 0.6c$. The charge at both events is located at z_e and has a corresponding Lorentz factor, $\gamma = 1.25$. The $z + ct = 0$ plane, in each case, denotes the causality limit of fields.

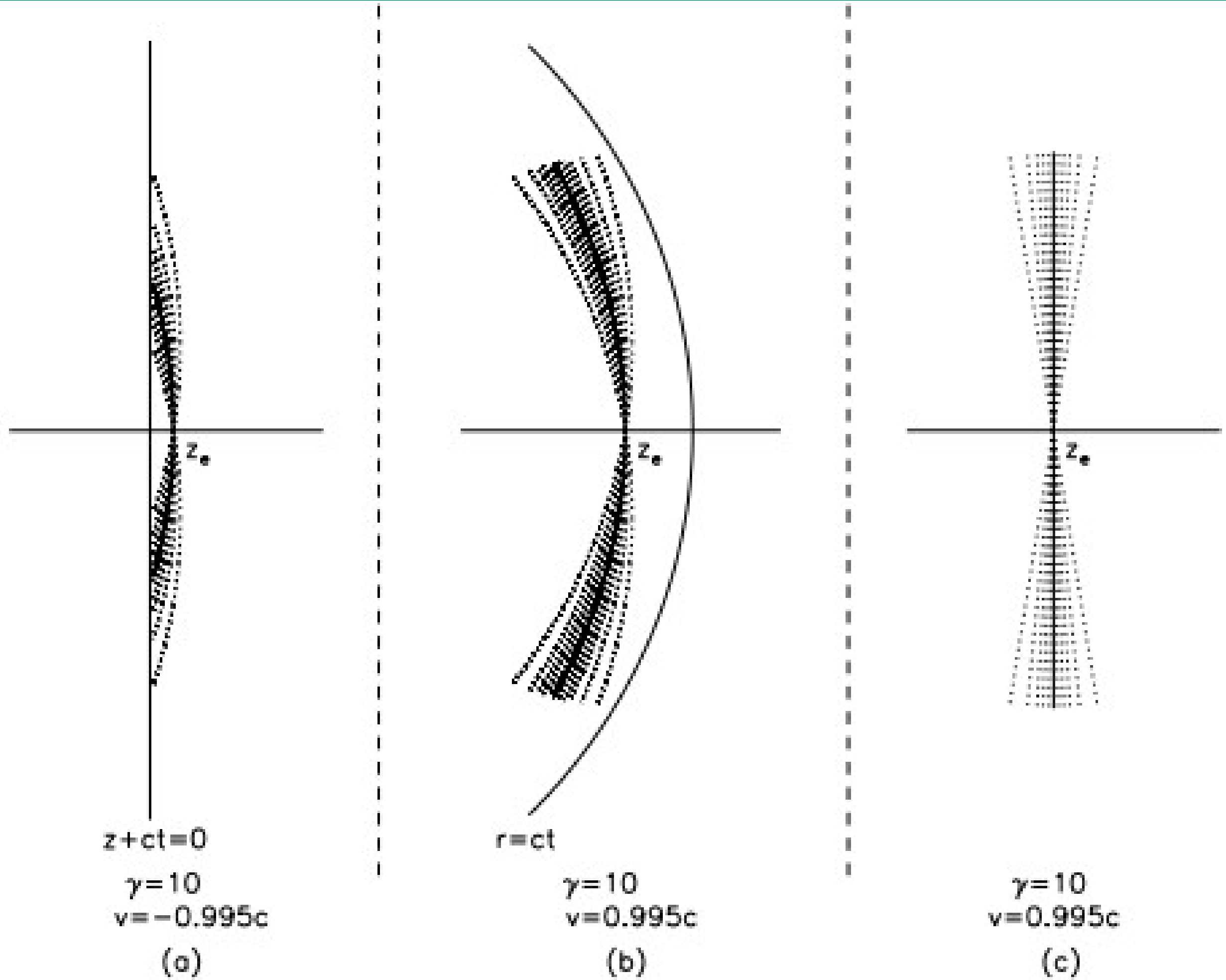

FIG. 2. The electric field distribution of a charge (a) moving with a uniform proper acceleration, and is presently at z_e moving with a 'present' velocity $v_0 = -0.995c$, corresponding to $\gamma_0 = 10$ (b) moving with a uniform proper acceleration, and is presently at z_e moving with a 'present' velocity $v_0 = 0.995c$, corresponding to $\gamma_0 = 10$ (c) moving with a uniform velocity $v = 0.995c$, corresponding to $\gamma = 10$. The spherical wave-front $r = ct$ is shown in the case of uniformly accelerated charge.

The electromagnetic fields, can be expressed in polar coordinates (R, ψ), centered on the instantaneous charge position, $z_e = \sqrt{z_0^2 + (ct)^2} = \gamma_0 z_0$.[3] Substituting $z = z_e + R\cos\psi$, $\rho = R\sin\psi$ and $v_0 = ct/z_e$ in Eq. (4), and after some algebraic manipulations, we get

$$E_R = \frac{e(1+\eta\cos\psi)}{R^2\gamma_0^2[1+\eta^2+2\eta\cos\psi-(v_0/c)^2\sin^2\psi]^{3/2}}$$
$$E_\psi = \frac{e\eta\sin\psi}{R^2\gamma_0^2[1+\eta^2+2\eta\cos\psi-(v_0/c)^2\sin^2\psi]^{3/2}}$$
$$B_\phi = \frac{e(v_0/c)\sin\psi}{R^2\gamma_0^2[1+\eta^2+2\eta\cos\psi-(v_0/c)^2\sin^2\psi]^{3/2}}, \tag{14}$$

with $\eta = aR/2\gamma_0 c^2$. All the remaining field components are zero.

The Poynting vector can be decomposed into radial and transverse components

$$S_{\rm R} = \frac{c}{4\pi} E_\psi B_\phi \tag{15}$$

$$S_\psi = \frac{-c}{4\pi} E_{\rm R} B_\phi \tag{16}$$

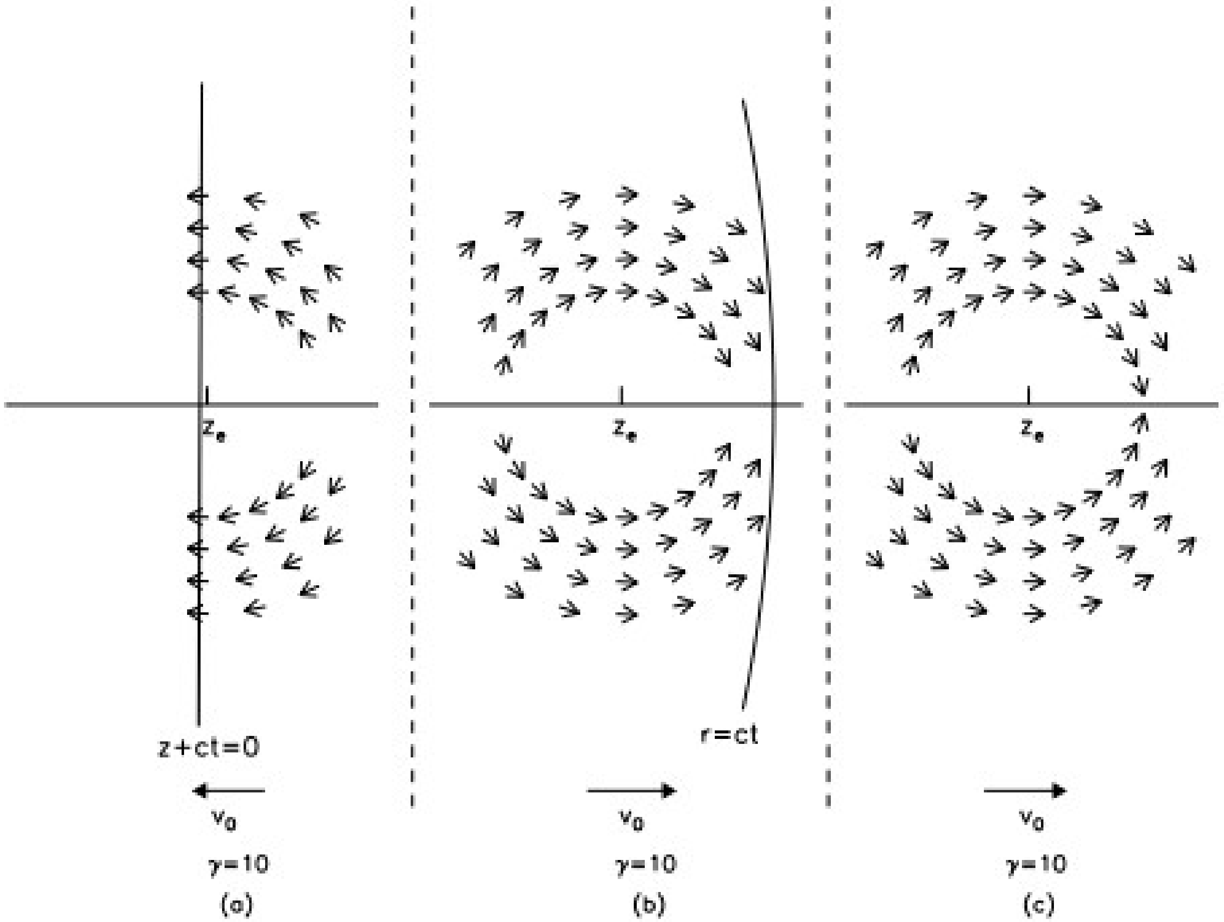

FIG. 3. The transverse Poynting vector component, S_ψ, for a charge (a) moving with a uniform proper acceleration, and is presently at z_e moving with a 'present' velocity $v_0 = -0.995c$, and actually getting decelerated (b) moving with a uniform proper acceleration, and is presently at z_e moving with a 'present' velocity $v_0 = 0.995c$, corresponding to $\gamma_0 = 10$ (c) moving with a uniform velocity $v = 0.995c$, corresponding to $\gamma = 10$. The spherical wave-front $r = ct$ is shown in the case of uniformly accelerated charge. The overall Poynting flow is along the direction of motion of the charge.

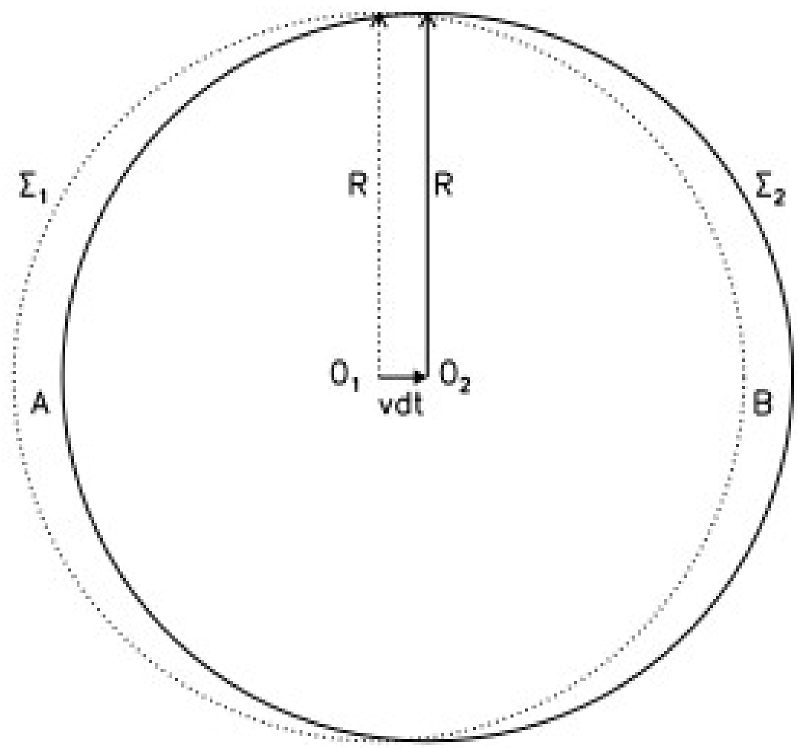

FIG. 4. As a charge moves with a velocity v from its position O_1 to O_2, its self-fields also move with it. If we consider two spherical volumes, Σ_1 and Σ_2, each of radius R around the two charge positions, the field energy in the region of intersection B between the two spheres increases at the cost of the field energy in the region A, where it reduces. The Poynting flows in Figures 2 as well as 5, represent the convective flow of the self-field energy along with the moving charge.

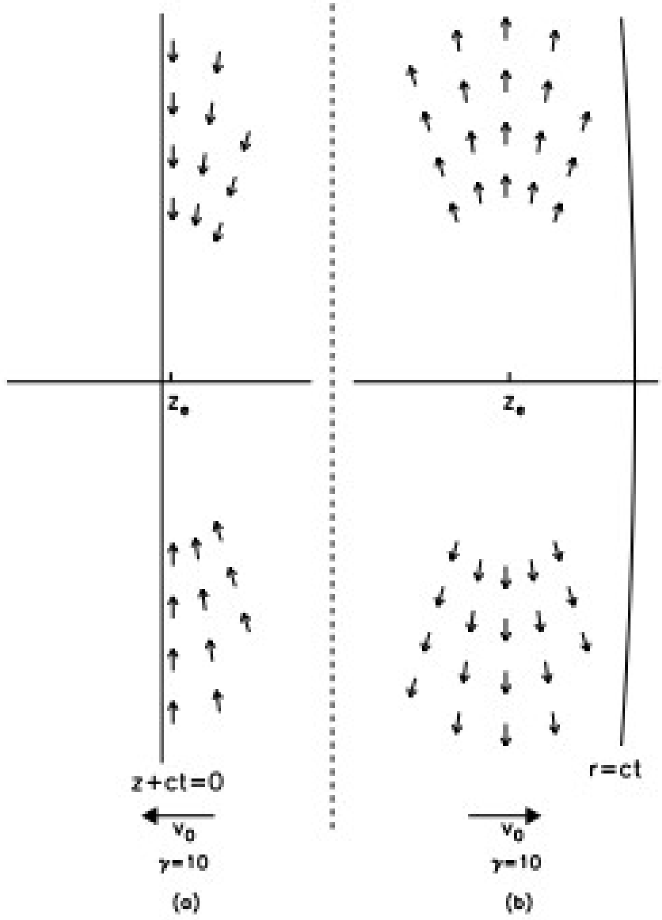

FIG. 5. The radial Poynting vector component, $S_{\rm R}$, (a) for a charge moving with a uniform proper acceleration, and is presently at z_e moving with a 'present' velocity $v_0 = -0.995c$, and actually getting decelerated with the Poynting flow being *everywhere* inward toward the charge (b) for a charge moving with a uniform proper acceleration, and is presently at z_e moving with a 'present' velocity $v_0 = 0.995c$, corresponding to $\gamma_0 = 10$, the Poynting flow being outward for the accelerating charge. The spherical wave-front $r = ct$ is shown with respect to z_e.

It is a misconception that the radiation emitted from the uniformly accelerated charge goes beyond the horizon, the regions of space-time inaccessible to an observer co-accelerating with charge. In fact, from Eq. (5), $E_\rho = 0$ at the $z = 0$ plane at any time t, implying that there is no component of Poynting flux through the $z = 0$ plane ever. This statement is true for all inertial frames at all times. The only exception is at $t = 0$ when an infinite z-component of Poynting vector due to δ-fields is present at $z = 0$. However, the δ-field is in no way causally related to the charge during its uniform acceleration, whose influence lies only in the $z + t > 0$ region only. All fields, originating from the accelerating charge positions, lie in the region $z > 0$ at time $t = 0$ and the radiation, if any, from the accelerating charge should also lie only be present there. In fact, it has been explicitly shown[1] that the δ-field has a causal relation with the event when acceleration is first imposed on the charge at infinity and because of a rate of change of acceleration at that event the charge moving undergoes radiation losses owing to the Lorentz-Dirac radiation reaction, that neatly explains the total energy lost by the charge into δ-field.[1]

In a typical radiation scenario, the radiation moves away (to infinity!), with the charge remaining behind, perhaps not very far from its original location. Such a bound motion of the charge necessarily implies its velocity and acceleration having a sort of cyclic behaviour.

However, in the case of a uniform acceleration, such is not true. First thing, as the charge picks up speed due to its constant acceleration, its Doppler factor soon becomes large and due to the relativistic beaming. the fields as well as the associated Poynting flux, lie in a narrow cone of an opening angle, $\theta \sim 1/\gamma$, around the direction of motion.

Secondly, the charge, moving with $v \to c$, is not very far behind the wave-front $r = ct$, and lags behind by a distance $\sim ct(1 - 1/2\gamma^2)$, and the fields are all around the charge 'present' position. Actual calculations show that the fields, in fact, very much resemble those of a charge moving with a uniform velocity equal to the 'present' velocity of the uniformly accelerated charge.

The rate of energy being "fed" into fields during the acceleration phase is exactly equal to that of the energy being "retrieved" from the fields during the deceleration phase. Why we see only an outward Poynting flow for $r \to \infty$ is because by the nature of the problem it also means $t \to \infty$, with the charge then being necessarily in the accelerating phase, and accordingly there being seen only an outgoing Poynting flux.

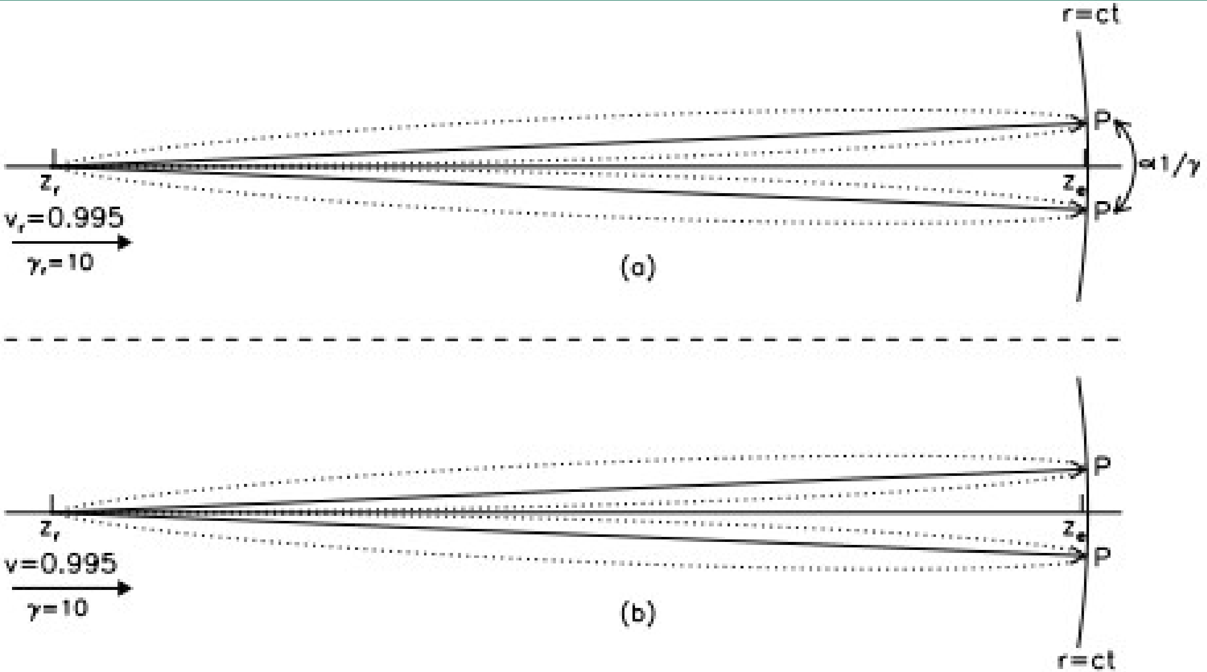

FIG. 6. Angular distribution of the electric field strength with respect to the time-retarded position z_r of the charge, moving along the z-axis with a velocity $v = 0.995$ and the corresponding Lorentz factor $\gamma = 10$. The maxima of the field strength is along the direction OP, which is at an angle $\theta = 1/\sqrt{5}\gamma$. Thus the field distribution mostly lies with a cone of angle $\theta \sim 1/\gamma$. By the time the fields from the retarded position z_r reach at the field point P, the charge moves to z_e. The maxima of the electric field with respect to 'present' position z_e of the charge seems to be in a plane perpendicular to the z-axis. The upper panel (a) is for the uniformly accelerated charge, while the lower panel (b) is for a charge moving with a uniform velocity. There is hardly any difference in the two panels, except that the 'present' charge position is slightly nearer to the spherical front in the case (a) where because of the acceleration the final velocity becomes $v = 0.99995$ than in case (b), where the velocity throughout remains unchanged at $v = 0.995$.

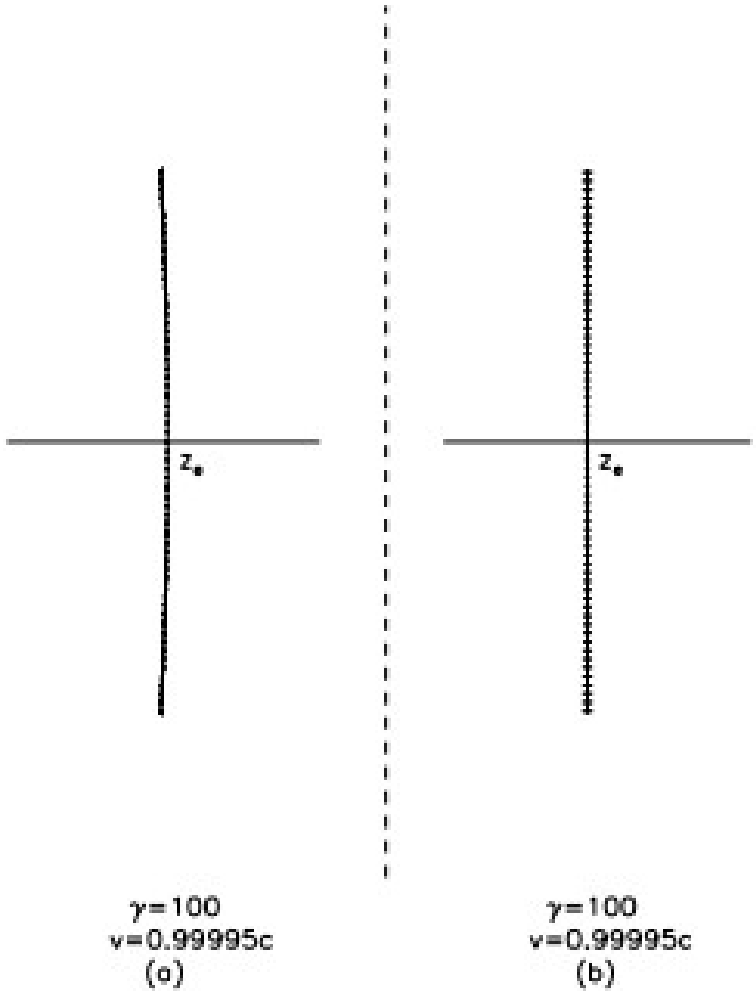

FIG. 7. The electric field distribution (a) of a uniformly accelerated charge, with a 'present' velocity $v = 0,99995c$, corresponding to $\gamma = 100$ (b) of a charge moving with a uniform velocity $v = 0,99995c$, corresponding to $\gamma = 100$.

It has been shown, from explicit calculations, that in the case of a charge moving with a uniform velocity v_0 and a corresponding Lorentz factor γ_0, the energy in the transverse self-fields in a region between two spheres of retarded radii r and $r + dr$ is,[3]

$$d\mathcal{E} = \frac{2e^2}{3c^2}(\gamma_0 v_0/c)^2\,\frac{dr}{r^2}\,. \tag{21}$$

Now, we consider a charge with uniform acceleration $a = \gamma^3\dot{v}$, instantly stationary (i.e., $v = 0$) at time $t = 0$, then the energy in its transverse fields is zero. After a time t, when it is moving with a velocity $\gamma_0 v_0 = at = ar/c$, due to its 'present' velocity v_0, an energy in its transverse self-fields in a spherical shell between r and $r + dr$ would be,

$$d\mathcal{E} = \frac{2e^2}{3c^2}(\gamma_0 v_0/c)^2\frac{dr}{r^2} = \frac{2e^2}{3c^3}a^2 dt\,, \tag{22}$$

the last term on the right hand side exactly accounting for the commonly–believed energy 'radiated away' during the time interval $dt = dr/c$, which actually goes into building the self-fields of the uniform accelerated charge.

After all for a charge now moving with $v_0 = 0.99995$ and $\gamma_0 = 100$, its self-field energy $\propto v_0^2\gamma_0^2 = 10^4$ was zero at $t = 0$ with $v_0 = 0.$ and $\gamma_0 = 1$ with minimal self-field energy. This growth in self-field energy could come where else from but the acceleration fields.

As the fields move toward infinity, so does the charge. The fields actually are the self-fields of the charge and as the charge picks up speed, the fields increase in strength, due to the acceleration fields, to an equivalent value expected from those of the charge moving with a uniform velocity which is equal to the 'present' velocity of uniformly accelerated charge.

At $t > 0$, the Poynting flux at all distances is radially away from the charge as it is getting accelerated, and the energy in the self-fields is increasing even at far-off points. On the other hand. at $t < 0$, when the charge is getting decelerated, the Poynting flux at all distances is radially inward toward the 'present' charge position, again at all distances from it. This is because the energy in the self-fields is decreasing even at far-off points. Of course at $t = 0$, when the charge is stationary, Poynting flux is zero everywhere.

The answer to the question why the comoving acceleration frame observer does not see any Poynting flux while in any inertial frame, sooner or later, an observer does find the Poynting flux, is that actually in the comoving acceleration frame, the observer is continuously jumping from one instantaneous *rest frame* to another, where transverse self-fields are nil.

One has to actually, at an given time, take a holistic view of the fields everywhere, i.e., at all distances from the charge, even if the fields at various points get determined from the corresponding time-retarded past positions of the charge. One comes across such a scenario in the case of a charge moving with a uniform velocity, where electric fields are in radial directions from the 'present' position of the charge, even though the fields everywhere are determined from the past time-retarded positions of the charge. The argument used there is that in the field expressions, since no acceleration term is being used, the information that gets fed into the field computation is that the charge is moving with a uniform velocity. The situation is similar in the case of a uniform acceleration case, where the field expressions determine fields according to the information of only the velocity and acceleration of the charge and there is no term in the field expressions for 'a rate of change of acceleration', the field expressions therefore determine fields for a farther point at distance r for time, $t = r/c$, for a uniform acceleration case.

Naturally there is no radiation reaction on the uniformly accelerated charge, since no field energy is being 'radiated away' from such a charge. This, of course, also makes it fully conversant with the strong principle of equivalence.

There is thus no radiation going away from the charge, instead the fields, including the contribution of acceleration fields, remains attached to the charge and is not dissociated from the charge as long as it is moving with a uniform acceleration.

The picture that emerges is this. In this case, the 'present' velocity of the charge is also constantly increasing, approaching c, and the Lorentz factor γ approaching ∞. Since the information about the velocity and the acceleration of the charge is contained in the electric field expression and from the value of acceleration the future velocity of the charge after a time t is known, the acceleration field in collusion with velocity fields tends to keep the energy in the causally linked spherical surface at $r = ct$ synchronized with the extrapolated velocity of the uniformly accelerated charge at time t. As long as the charge continues to move with the same value of (uniform) acceleration, the things remain synchronized as far the total energy in the self-fields is concerned. Only when there is a change in the acceleration, and as there is no information contained in the field expressions about the rate of change of acceleration, then the energy in fields does not remain synchronized with that of the actual motion of the charge, which is affected by the rate of acceleration, therefore there is a mismatch in field energies and the excess energy becomes the radiated away energy, no longer part of the self-fields of the charge, and thus dissociated from the charge.

The contrary conclusions in literature have been arrived at in most cases, firstly, by considering only the acceleration fields, an approach which though might be valid in vast majority of cases of radiation from an accelerated charge, is not valid in the case of a uniformly accelerated charge. This is because in this case the velocity at the retarded time, $v_{\rm r} \propto at = ar/c$ and then the velocity fields, $v/r^2 \propto ar/cr^2 = a/rc$ become comparable to the acceleration fields $\propto a/rc$ for all r.

Secondly, almost no attention is generally paid to the 'present' position of the charge which during the intervening time interval $t = r/c$, when the fields move to a large r (toward infinity!) is almost keeping in step with the fields, being just a distance $\sim r(1-1/\gamma^2)$ behind for all r, and with γ continuously increasing due to the uniform acceleration. In fact, fields remain appreciable only in a region $\Delta z \propto 1/\gamma$, very similar to the uniform velocity case where electric field is appreciable near the 'present' position of the charge, in a region whose extent falls as $1/\gamma$ and where the fields component is mostly along the direction normal to the direction of motion.

References

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[3] A. K. Singal, "The equivalence principle and an electric charge in a gravitational field II. A uniformly accelerated charge does not radiate," Gen. Rel. Grav. **29**, 1371-1390 (1997).